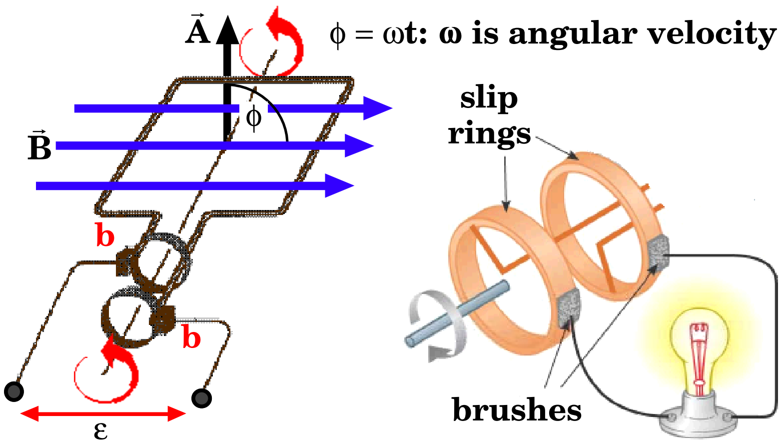
CHAPTER 29

ALTERNATING CURRENT CIRCUITS

- **ac generators**
- **Resistance in ac circuits and rms values**
  - † *Introduction to phasors*
- **Inductors in ac circuits**
- **Capacitors in ac circuits**
- ***R-C-L circuits with an ac generator***
  - † *Resonance and tuned circuits*

$\phi = \omega t$: **ω is angular velocity**

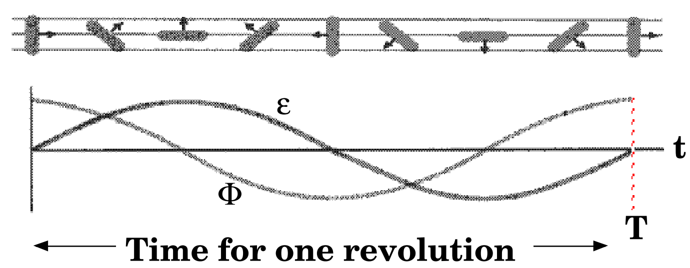


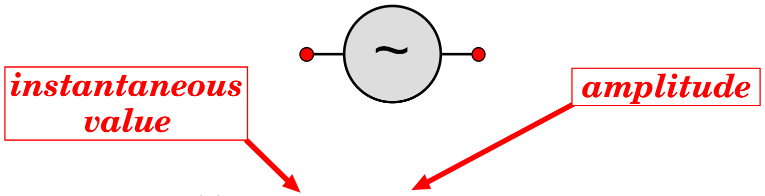
Flux through coil:

$$\Phi = \vec{B} \bullet \vec{A} = BA\cos\phi = BA\cos(\omega t).$$

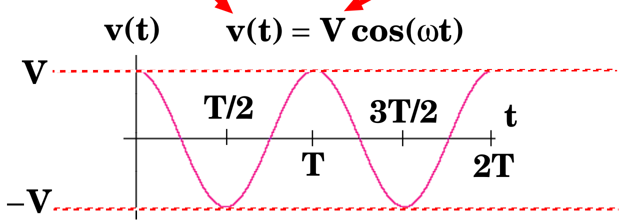
$$\therefore \frac{d\Phi}{dt} = -\omega BA\sin(\omega t).$$

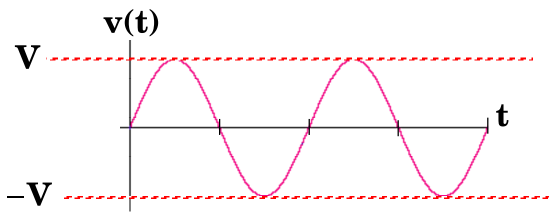
$$\therefore \varepsilon = -\frac{d\Phi}{dt} = \omega BA\sin(\omega t).$$





$$v(t) = V\cos(\omega t)$$





$$v(t) = V\sin(\omega t)$$

$$\text{Frequency } f = \frac{\omega}{2\pi}$$

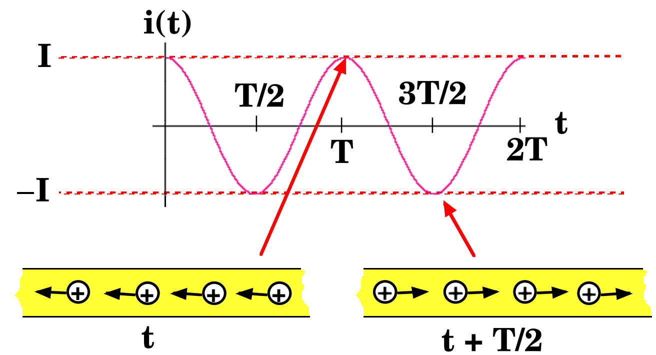
If $\omega \Rightarrow$ rad/s then $f \Rightarrow$ Hertz (Hz).

If $f = 60$ Hz, $\omega = 376$ rad/s

$$\text{Periodic time} = T = \frac{1}{f} = 16.7 \text{ ms.}$$

Alternating p.d. $\Rightarrow$ alternating current:

$$i(t) = i = I\cos(\omega t)$$



With household a.c. (60Hz) the current reverses direction every 1/120 second, i.e., 120 times every second!

$$\therefore T = 1/f = 1.67 \times 10^{-2}\text{s.}$$

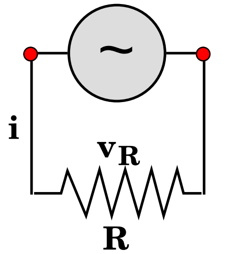
Distance travelled in one-half cycle:

$$\sim v_D \times T/2 \Rightarrow 10^{-4} \times 1.67 \times 10^{-2}/2$$

$$\approx 8 \times 10^{-7}\text{m.}$$

(~ 3000 atom diameters)

Not very far!!

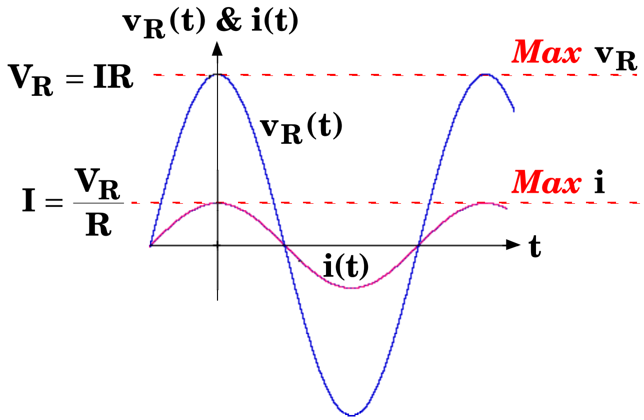


Instantaneous **current:** $i(t) = i = I\cos(\omega t)$

Instantaneous **potential difference across R:**

$$v_R = iR = IR\cos(\omega t) = V_R\cos(\omega t)$$

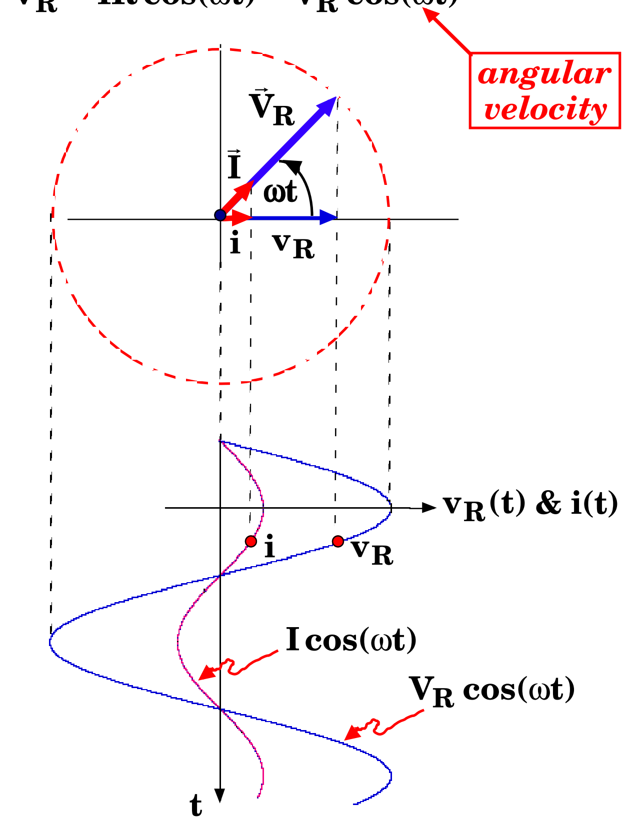
(which is the same as the emf of the generator)



The p.d. and current are *in-phase*.

$$i = I\cos(\omega t)$$

$$v_R = IR\cos(\omega t) = V_R\cos(\omega t)$$



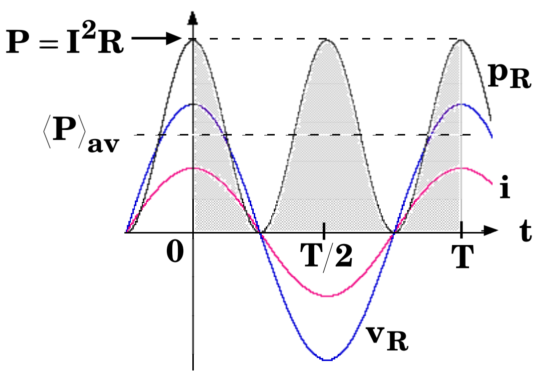
A "Phasor" is a rotating vector. They can be added just like ordinary vectors.

Power dissipated in a resistor:

The ***instantaneous*** power $p_R = v_R i$.

But $i = I\cos(\omega t)$ and $v_R = IR\cos(\omega t)$

$$\therefore p_R = I^2 R\cos^2(\omega t)$$



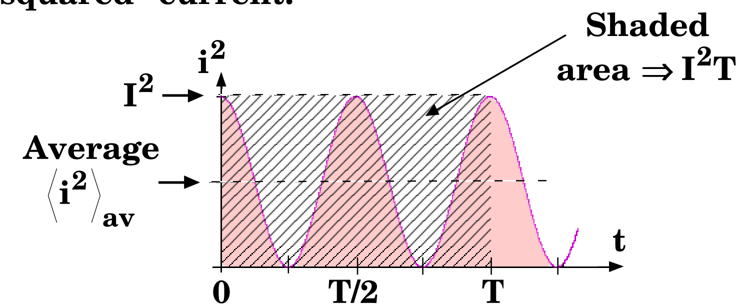
The *average power* (over one period):

$$\langle P\rangle_{av} = \frac{\text{Energy dissipated in one period}}{\text{time of one period}}$$

$$= \frac{1}{T}\int_0^T I^2 R\cos^2(\omega t)\,dt = \frac{1}{2}I^2 R$$

i.e., ***half the maximum power***.

Let's look for a moment at $i^2 = I^2\cos^2(\omega t)$, the "squared" current.



Average "squared" current over one cycle is:

$$\langle i^2\rangle_{av} = \frac{1}{T}\int_0^T I^2\cos^2(\omega t)\,dt = \frac{1}{2}I^2.$$

Define ***the root mean square (rms)*** current as:

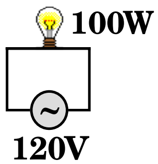
$$I_{rms} = \sqrt{\langle i^2\rangle_{av}} = \sqrt{\frac{1}{2}I^2} = \frac{I}{\sqrt{2}} \quad (= 0.707I)$$

... and similarly: $V_{rms} = \dfrac{V}{\sqrt{2}} \quad (= 0.707V)$.

From before, average power: $\langle P\rangle_{av} = \dfrac{1}{2}I^2 R$

$$= I_{rms}^2 R \qquad \left(= V_{rms}I_{rms} = \frac{V_{rms}^2}{R}\right)$$

Problem 29.23 page 963:



Given : $\langle P\rangle_{av} = 100W$ and $V_{rms} = 120V$.

But $\langle P\rangle_{av} = V_{rms}I_{rms}$

$$\therefore I_{rms} = \frac{100}{120} = 0.833A.$$

Also $I_{rms} = \frac{I_{max}}{\sqrt{2}}$ and $V_{rms} = \frac{V_{max}}{\sqrt{2}}$

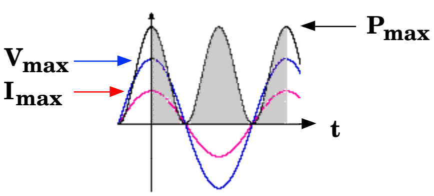
$$\therefore I_{max} = \sqrt{2} \times I_{rms} = 1.18A$$

and $V_{max} = \sqrt{2} \times V_{rms} = 169.7V$.

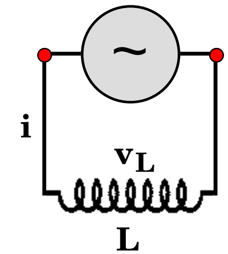
The *maximum* (instantaneous) power is:

$$P_{max} = I_{max}V_{max} = 1.18 \times 169.7 = 200W,$$

i.e., *maximum power* = *twice average power*.



Note: the power pulses occur at 2ω.



Instantaneous current: $i = I\cos(\omega t)$

Instantaneous p.d. across inductor:

$$v_L = L\frac{di}{dt} = -I\omega L\sin(\omega t)$$

(which is the same as the emf of the generator.)

But $-\sin(\omega t) = \cos(\omega t + \pi/2)$ ... *check it out!*

$$\therefore v_L = I\omega L\cos(\omega t + \pi/2)$$

$X_L = \omega L$ *"reactance"* (Ω) *"phase angle"*

Reactance (X_L) is like "*ac resistance*" but it is not constant; *it depends on frequency*.

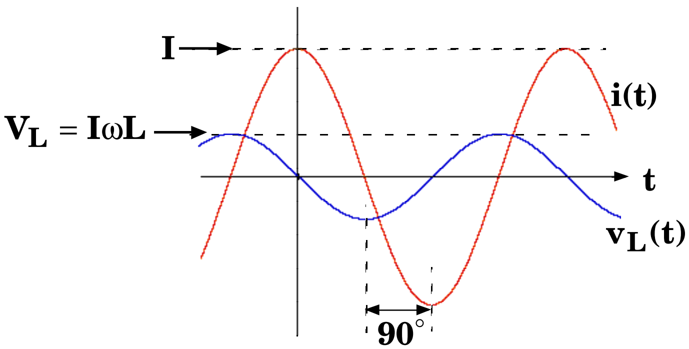
i.e., $v_L = IX_L\cos(\omega t + \pi/2)$

Therefore, in terms of angle, v_L is 90° *ahead* of the current!

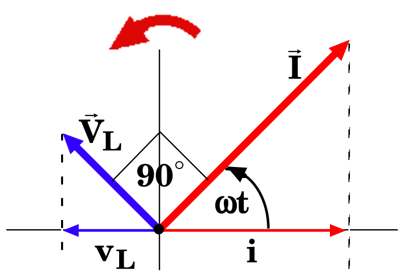
$$i = I\cos(\omega t)$$

$$v_L = IX_L \cos(\omega t + 90^\circ) = V_L \cos(\omega t + 90^\circ)$$

i.e., $V_L = IX_L \quad (= I\omega L)$



v_L is 1/4 of a cycle (90°) *ahead* of i



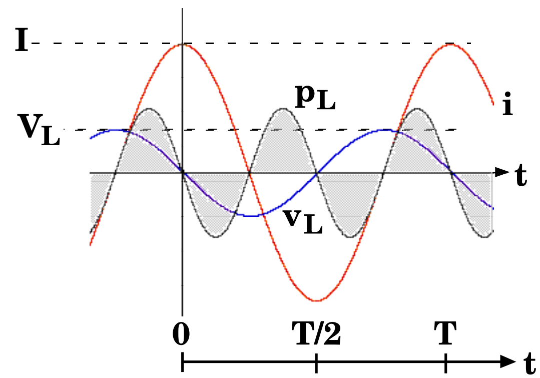
Phasor diagram

Power in an ideal inductor in an a.c. circuit:

The *instantaneous* power $p_L = v_L i$.

But $i = I\cos(\omega t)$ and $v_L = I\omega L\cos(\omega t + 90^\circ)$

$$\therefore p_L = I^2\omega L\cos(\omega t)\cos(\omega t + 90^\circ)$$

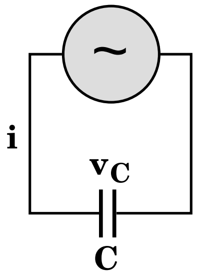


The average power (over one cycle):

$$\langle P\rangle_{av} = \frac{1}{T}\int_0^T I^2\omega L\cos(\omega t)\cos(\omega t + 90^\circ)\,dt = 0.$$

... You can tell that from the graph! ...

Energy stored in inductor in one quarter-cycle is released in the next quarter-cycle.



Instantaneous **current:**

$$i = I\cos(\omega t)\ \left(= \frac{dq}{dt}\right)$$

Instantaneous **charge on the capacitor:**

$$q = \int i\,dt = I\int \cos(\omega t)\,dt = \frac{I}{\omega}\sin(\omega t).$$

$\therefore$ ***instantaneous*** **p.d. across the capacitor is:**

$$v_C = \frac{q}{C} = \frac{I}{\omega C}\sin(\omega t),$$

(which is the same as the emf of the generator.)

But $\sin(\omega t) = \cos(\omega t - \pi/2)$,

so $v_C = \frac{I}{\omega C}\cos(\omega t - \pi/2)$.

$X_C = \frac{1}{\omega C}$ ***reactance*** (Ω) ***phase angle***

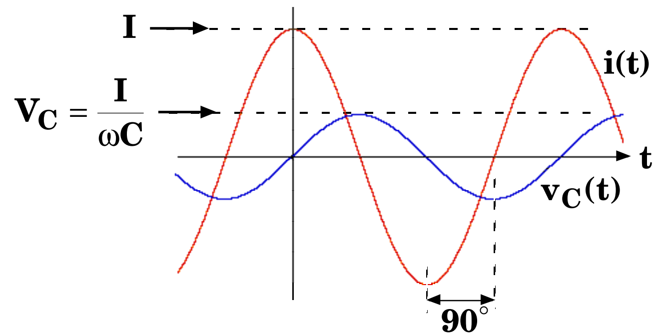
i.e., $v_C = IX_C\cos(\omega t - \pi/2)$

Therefore, in terms of angle, v_C is 90° *behind* the current.

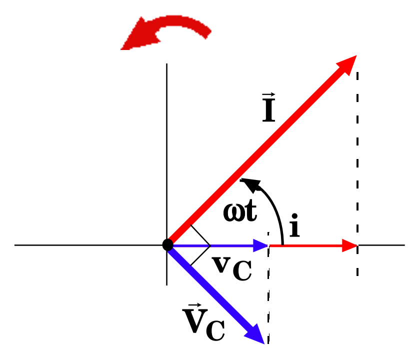
$$i = I\cos(\omega t)$$

$$v_C = IX_C\cos(\omega t - \pi/2) = V_C\cos(\omega t - \pi/2)$$

$$\text{i.e., } V_C = IX_C \qquad \left(= \frac{I}{\omega C}\right)$$



v_C is 1/4 of a cycle (–90°) *behind* i



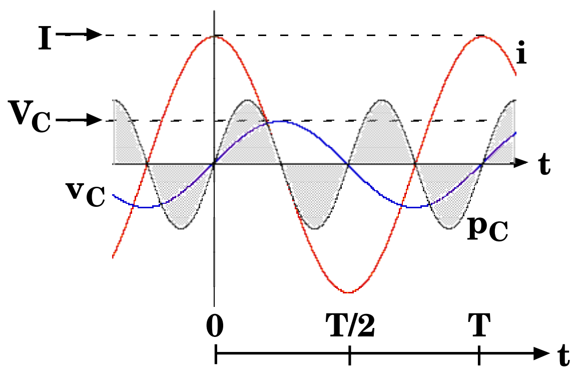
Phasor diagram

Power in a capacitor in an a.c. circuit:

The *instantaneous* power $p_C = v_C i$.

But $i = I\cos(\omega t)$ and $v_C = \frac{I}{\omega C}\cos(\omega t - 90^\circ)$

$$\therefore p_C = \frac{I^2}{\omega C}\cos(\omega t)\cos(\omega t - 90^\circ)$$



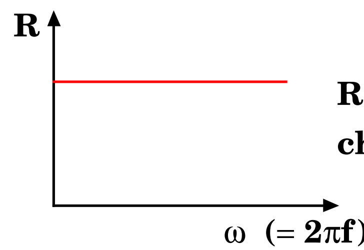
The average power (over one cycle):

$$\langle P\rangle_{av} = \frac{1}{T}\int_0^T \frac{I^2}{\omega C}\cos(\omega t)\cos(\omega t - 90^\circ)\,dt = 0.$$

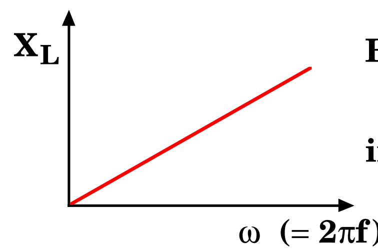
... look at graph!! ...

Energy stored in the capacitor in one quarter-cycle is released in the next quarter-cycle.

Comparison of resistance and reactance ...



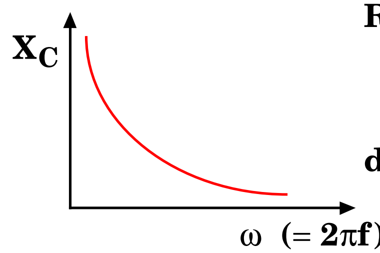
Resistance does not change with frequency



Reactance of an inductor

X_L (= ωL)

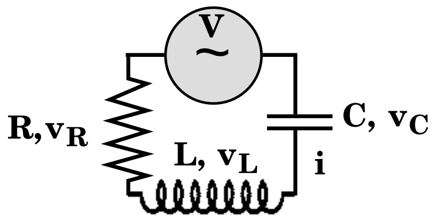
increases with frequency

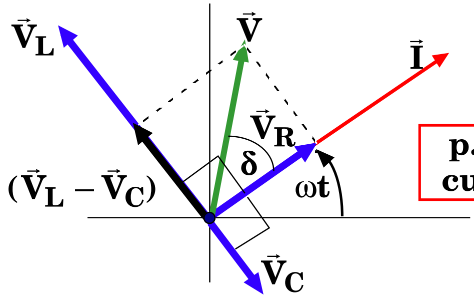


Reactance of a capacitor

X_C (= $\frac{1}{\omega C}$)

decreases with frequency

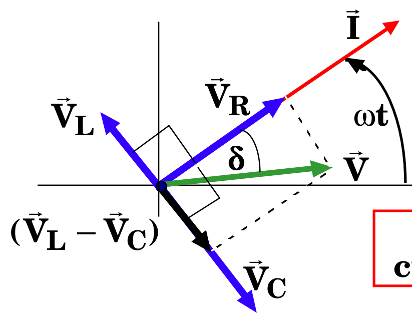




$X_L > X_C$

p.d. ***LEADS*** current by δ

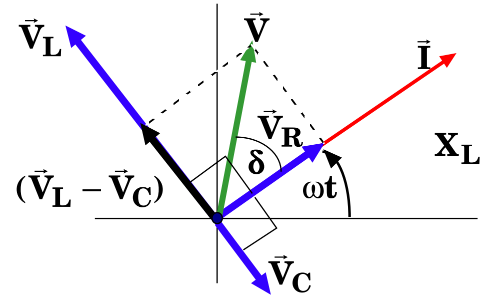
By Kirchoff's 2nd rule: *the resultant p.d. supplied the generator* $\vec{V} = \vec{V}_R + (\vec{V}_L + \vec{V}_C)$.



$X_L < X_C$

p.d. ***LAGS*** current by δ

The angle δ is called the *phase angle*.



$X_L > X_C$

$$V = |\vec{V}| = \sqrt{(V_R)^2 + (V_L - V_C)^2}$$

$$= I\sqrt{R^2 + (X_L - X_C)^2} = IZ \quad \Rightarrow \text{"Ohms's law"}$$

where Z is the *impedance*, given by:

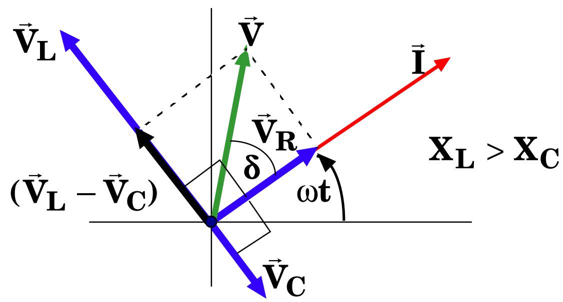
$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \Rightarrow \quad (\text{units } \Omega).$$

Z is like the "*ac resistance*" of the circuit. Note: $Z \Rightarrow Z(\omega)$. The *instantaneous* p.d. is

$$v = V\cos(\omega t + \delta) = IZ\cos(\omega t + \delta)$$

where $\delta = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$

$$= \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right).$$



The *instantaneous* power is:

$$P = vi = V\cos(\omega t + \delta) \times I\cos(\omega t)$$

$$= VI\cos^2(\omega t)\cos\delta + \cos(\omega t)\sin(\omega t)\sin\delta$$

$$= VI\cos^2(\omega t)\cos\delta + \frac{1}{2}\sin(2\omega t)\sin\delta$$

If we average over one cycle we have:

$$\left\langle \cos^2(\omega t) \right\rangle_{av} = \frac{1}{2} : \quad \left\langle \sin(2\omega t) \right\rangle_{av} = 0$$

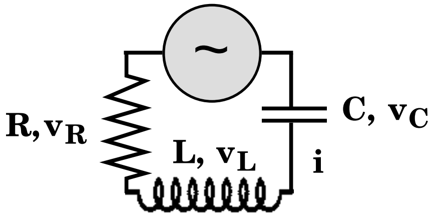
(so second term $\Rightarrow 0$)

$$\therefore \langle P \rangle_{av} = \frac{1}{2} VI\cos\delta$$

$$= V_{rms} I_{rms} \cos\delta$$

power factor

In summary ...



The amplitude of the potential difference across generator (emf) is:

$$V = IZ$$

where V is the magnitude of the phasor:

$$\vec{V} = \vec{V}_R + \vec{V}_L + \vec{V}_C$$

(not an algebraic sum!) and Z is the impedance, given by:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

So, we have relationships between V, I and Z:

$$V = IZ \qquad I = \frac{V}{Z} \text{ and } Z = \frac{V}{I}$$

... that are the "*a.c. equivalent*" of Ohm's law.

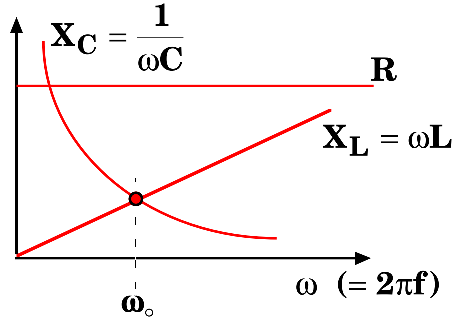
Since $Z = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$,

then Z is a ***minimum*** at some $\omega = \omega_\circ$ when

$$\omega_\circ L = \frac{1}{\omega_\circ C},$$

i.e., when $X_L = X_C$.

Then $\omega_\circ = \sqrt{\dfrac{1}{LC}} \Rightarrow$ ***resonant frequency***.



At resonance, $\omega = \omega_\circ$,

$$\therefore Z = R,$$

i.e., the impedance is purely *resistive*.

Problem 29-66 page 966:

Given : $L = 1\mu H$: $f_1 = 500 kHz$: $f_2 = 1600 kHz$.

At resonance: $\omega_\circ L = \dfrac{1}{\omega_\circ C}$. $\therefore C = \dfrac{1}{\omega_\circ{}^2 L}$

But $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$, so

$$C_1 = \frac{1}{(2\pi \times 500 \times 10^3)^2 \times 1 \times 10^{-6}}$$

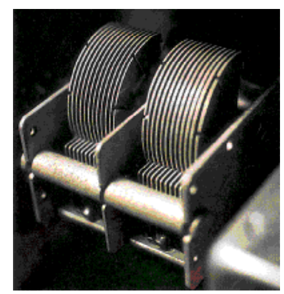
$$= 101 \times 10^{-9} F = 101 nF,$$

and

$$C_2 = \frac{1}{(2\pi \times 1600 \times 10^3)^2 \times 1 \times 10^{-6}}$$

$$= 9.89 \times 10^{-9} F = 9.89 nF.$$

Note that the result does not depend on R.



Variable capacitor

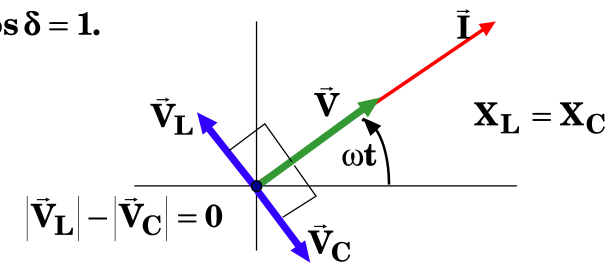
What's the power factor and the average power at resonance?

We know: $\langle P \rangle_{av} = V_{rms} I_{rms} \cos\delta$

power factor

where δ is phase angle between current and p.d. But, at resonance, $\delta = 0$, i.e., the current and p.d. are in-phase, since $X_L = X_C$.

$\therefore \cos\delta = 1.$

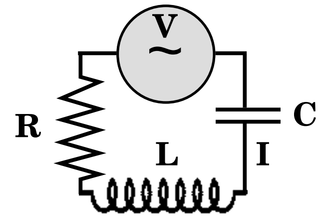


So, $\langle P(\omega_\circ) \rangle_{av} = V_{rms} I_{rms}$, **i.e., a maximum.**

Also, at resonance: $V_{rms} = I_{rms} R.$

$\therefore \langle P(\omega_\circ) \rangle_{av} = V_{rms}^2 / R = I_{rms}^2 R.$

How does the current vary with frequency? Take a generator with a constant voltage amplitude.



We have: $$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{V}{\sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega LC}\right)^2}} = \frac{V}{\sqrt{R^2 + L^2\left(\omega - \frac{\omega_\circ^2}{\omega}\right)^2}},$$

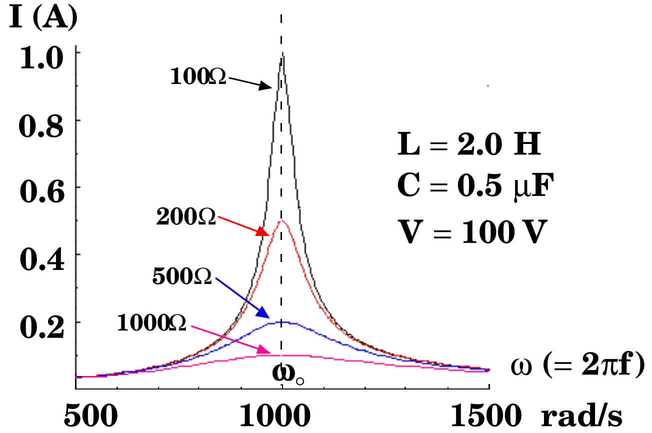
i.e., $$I = \frac{V\omega}{\sqrt{\omega^2 R^2 + L^2\left(\omega^2 - \omega_\circ^2\right)^2}}.$$

Note, when $\omega = \omega_\circ$, $I = \frac{V}{R} \Rightarrow$ ***pure resistance***

i.e., at resonance, the current is determined only by R (and it's a maximum).

What does this function look like?

$$I = \frac{V\omega}{\sqrt{\omega^2 R^2 + L^2\left(\omega^2 - \omega_\circ^2\right)^2}}$$



$$\omega_\circ = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{2.0 \times 0.5 \times 10^{-6}}} = 1000 \text{ rad/s,}$$

and independent of R.

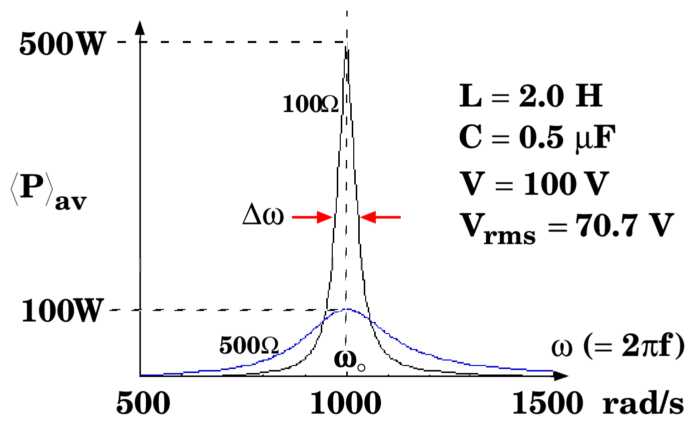
$$\therefore f = \frac{\omega_\circ}{2\pi} = 159.2 \text{ Hz.}$$

Note: $I_{max} = \frac{V}{R}$, **when** $\omega = \omega_\circ$.

How does the power vary with frequency?

$$\langle P\rangle_{av} = I_{rms}{}^2 R = \frac{V_{rms}{}^2\omega^2 R}{\omega^2 R^2 + L^2\left(\omega^2 - \omega_\circ^2\right)^2}.$$

Peak power: $P_{av}(\omega = \omega_\circ) \Rightarrow V_{rms}{}^2/R$



Q-value of the circuit ...

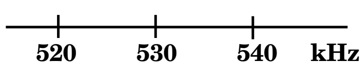
$$Q = \frac{\omega_\circ L}{R} \approx \frac{\omega_\circ}{\Delta\omega} = \frac{f_\circ}{\Delta f},$$

where $\Delta\omega$ (Δf) **is the width of the peak at half-maximum.**

Larger Q $\Rightarrow$ **sharper peak.**

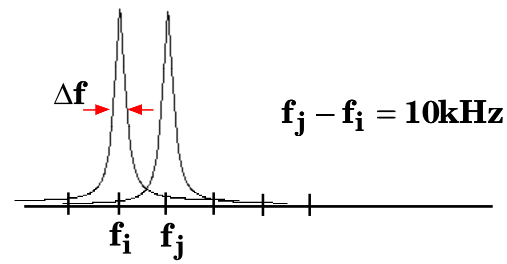
DISCUSSION PROBLEM [29.1] :

The frequency difference between radio stations on the AM band is 10kHz. What are suitable Q values at the lowest AM frequency (500kHz) and the highest AM frequency (1600kHz)?



DISCUSSION PROBLEM [29.1] : *...solution ...*

The power curves for the two neighboring stations (f_i and f_j) should be well separated ...



... so that only one of them is "detected". Make the width of the power functions (Δf) about $1/5$ th (i.e., 20%) of the channel separation,

$$\text{i.e.,} \quad \Delta f \sim 0.2(f_j - f_i) = 2\text{kHz}.$$

∴ At the low frequency end:

$$Q = \frac{f}{\Delta f} = \frac{500\text{kHz}}{2\text{kHz}} = 250.$$

∴ At the high frequency end:

$$Q = \frac{f}{\Delta f} = \frac{1600\text{kHz}}{2\text{kHz}} = 800.$$

Therefore, we need a Q value > 800.

Problem 29-87 page 967:

$I_{rms} = 10.0 A: \langle P \rangle_{av} = 720 W: V_{rms} = 120 V.$

(a) By definition: $V = IZ$ i.e., $V_{rms} = I_{rms} Z$.

$\therefore Z = V_{rms}/I_{rms} = 120/10 = 12.0 \Omega.$

(b) Also, $\langle P \rangle_{av} = I_{rms}{}^2 R$ ... ***why only R??***

where R is the resistance of the "device".

$\therefore R = \langle P \rangle_{av} / I_{rms}{}^2 = 7.20 \Omega.$

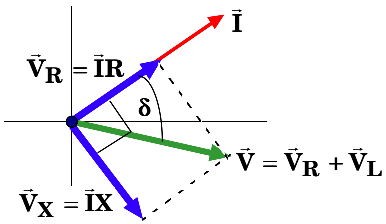
From notes we have: $Z = \sqrt{R^2 + (X_L - X_C)^2}$,

$\therefore Z^2 = R^2 + (X_L - X_C)^2 \Rightarrow R^2 + X^2$,

where X is the reactance of the "device".

$\therefore X = \sqrt{Z^2 - R^2} = \sqrt{12.0^2 - 7.20^2} = 9.60 \Omega.$

(c) If the current *leads* the p.d., the device is a (non-ideal) capacitor.



Extra ... from notes:

$\delta = \tan^{-1}(X/R) = 53.1^\circ$