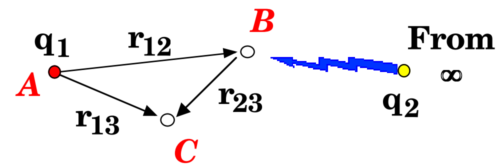
CHAPTER 24

ELECTROSTATIC ENERGY and CAPACITANCE

- **Electrostatic potential energy**
  - † ***Collection of charges***
- **Capacitance and capacitors**
  - † ***Storage of electrical energy***
- **Energy density of an electric field**
- **Combinations of capacitors**
  - † ***In parallel***
  - † ***In series***
- **Dielectrics**
  - † ***Effects of dielectrics***
- **Examples of capacitors**

***Electrostatic potential energy of a collection of charges*:**



The potential energy of an ensemble of charges is equal to the work done in bringing the charges together. Assume charge q_1 is in position *A* and bring in charge q_2 to *B* from ∞. The potential at *B* due to a charge q_1 at *A* is:

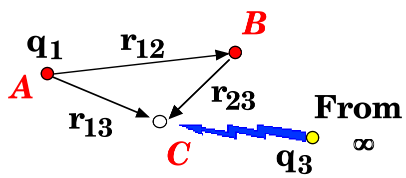
$$V_B = k\frac{q_1}{r_{12}}.$$

So, the work done (by you) in bringing a charge q_2 from ∞ to *B* is:

$$W_2 = -q_2\Delta V = -q_2(V_\infty - V_B) = -q_2(0 - V_B)$$

$$= k\frac{q_1 q_2}{r_{12}}.$$

Electrostatic potential energy of a collection of charges: *(continued)*



The work done in bringing charge q_3 from ∞ to *C* is:

$$W_3 = -q_3(V_\infty - V_C),$$

where V_C is the potential at *C* due to the charges at *A* and *B*.

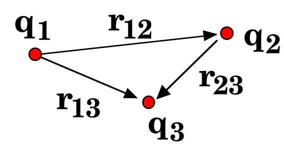
$$\therefore W_3 = -q_3\left(0 - (k\frac{q_1}{r_{13}} + k\frac{q_2}{r_{23}})\right) = k\frac{q_1q_3}{r_{13}} + k\frac{q_2q_3}{r_{23}}.$$

So, the <u>total</u> work done $= W_2 + W_3$

$$= k\frac{q_1q_2}{r_{12}} + k\frac{q_1q_3}{r_{13}} + k\frac{q_2q_3}{r_{23}}.$$

This is the ***electrostatic potential energy***, U, of to the three charges at *A*, *B* and *C*, i.e., the total work done assembling the charges.

Electrostatic potential energy of a collection of charges: *(continued)*



So, the ***electrostatic potential energy***, U, of a system of point charges is the total work done to assemble the charges from ∞ to their final positions. Now,

$$U = k\frac{q_1q_2}{r_{12}} + k\frac{q_1q_3}{r_{13}} + k\frac{q_2q_3}{r_{23}}$$

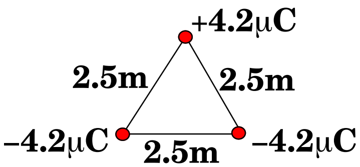
$$= \frac{1}{2}q_1\left(k\frac{q_2}{r_{12}} + k\frac{q_3}{r_{13}}\right) + \frac{1}{2}q_2\left(k\frac{q_3}{r_{23}} + k\frac{q_1}{r_{12}}\right)$$

$$+\frac{1}{2}q_3\left(k\frac{q_1}{r_{13}} + k\frac{q_2}{r_{23}}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{N} q_iV_i,$$

where V_i is the potential at the position of the ith charge due to all the other charges.

Problem 24-18(c) page 778:



The ***electrostatic potential energy***: $U = \frac{1}{2}\sum_{i=1}^{3} q_i V_i$

$$= \frac{1}{2}q_1\left[k\frac{q_2}{r_{12}} + k\frac{q_3}{r_{13}}\right] + \frac{1}{2}q_2\left[k\frac{q_3}{r_{23}} + k\frac{q_1}{r_{12}}\right] + \frac{1}{2}q_3\left[k\frac{q_1}{r_{13}} + k\frac{q_2}{r_{23}}\right]$$

$$= \frac{1}{2}(-4.2\times10^{-6})\times(9\times10^{9})\left[\frac{(-4.2\times10^{-6})}{2.5} + \frac{(+4.2\times10^{-6})}{2.5}\right]$$

$$+\frac{1}{2}(-4.2\times10^{-6})\times(9\times10^{9})\left[\frac{(+4.2\times10^{-6})}{2.5} + \frac{(-4.2\times10^{-6})}{2.5}\right]$$

$$+\frac{1}{2}(+4.2\times10^{-6})\times(9\times10^{9})\left[\frac{(-4.2\times10^{-6})}{2.5} + \frac{(-4.2\times10^{-6})}{2.5}\right]$$

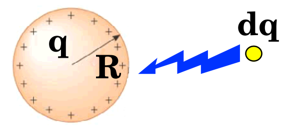
$$= (+4.2\times10^{-6})\times(9\times10^{9})\frac{(-4.2\times10^{-6})}{2.5}$$

$$= -0.0635\text{J}$$

?

What does the negative sign mean?

The electrostatic potential energy of a charged conducting sphere is equal to the amount of work *we do* in putting the charge onto the sphere. If the sphere already has a charge q, the work done in bringing a charge dq from ∞ onto the sphere is $dW = -dq(V_\infty - V)$, where V is the potential of the sphere. But $V_\infty = 0$ and $V = k\frac{q}{R}$,



$$\therefore dW = k\frac{q}{R}dq.$$

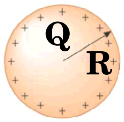
So, the total work done to charge the sphere from 0 → Q is:

$$W = \int_0^Q k\frac{q}{R}dq = \frac{k}{R}\left[\frac{q^2}{2}\right]_0^Q = \frac{1}{2}k\frac{Q^2}{R} = \frac{1}{2}QV,$$

where V (= kQ/R) is the potential of the fully charged sphere.

$$\therefore U = W = \frac{1}{2}QV.$$

Problem 24.19 page 778:



$$V = k\frac{Q}{R}$$

We've just shown that the electrostatic potential energy of a charged sphere is:

$$U = \frac{1}{2}QV.$$

But, by re-arrangement, we find $Q = \frac{VR}{k}$.

$$\therefore U = \frac{1}{2}\left(\frac{VR}{k}\right)V = \frac{V^2R}{2k}$$

$$= \frac{(2\times 10^3)^2 \times 0.1}{2\times 9\times 10^9} = 2.22\times 10^{-5}\,\mathrm{J}.$$

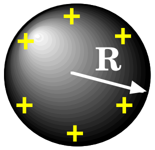
This is the amount of work we do in charging the sphere to a potential of 2kV.

- What is the charge on the sphere?

$$Q = \frac{RV}{k} = \frac{0.1\times 2\times 10^3}{9\times 10^9} = 2.22\times 10^{-8}\,\mathrm{C} \quad (22.2\mathrm{nC}).$$

CAPACITANCE:

When an object has a charge Q, it will have a potential V (= U/Q), because work is done (= U) to assemble the charge. (Conversely, if an object has a potential V it will have a charge Q.) The ***capacitance*** (C) of the object is the ratio Q/V.



Example: A charged spherical conductor carrying a charge Q. The potential of the sphere is

$$V = k\frac{Q}{R}.$$

Therefore, the ***capacitance*** of the sphere is

$$C = \frac{Q}{V} = \frac{Q}{kQ/R} = \frac{R}{k} = 4\pi\varepsilon_\circ R$$

UNITS: Capacitance $\Rightarrow$ Coulombs/Volts

$\Rightarrow$ Farad (F).

Example: A sphere with R = 10 cm (= 0.1m)

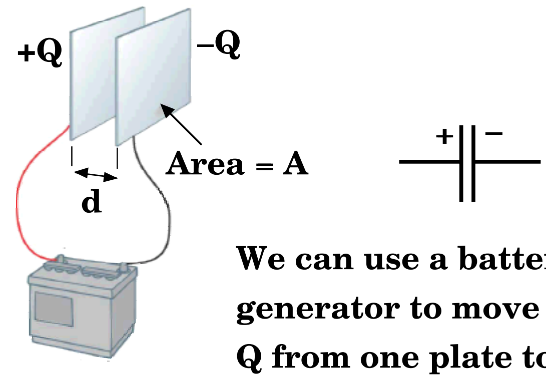
$C = 1.11\times 10^{-12}\,\mathrm{F}$ (= 1.11pF).

Capacitance is a measure of the "capacity" that an object has for charge, i.e., given two objects at the same potential, the one with the *greater capacitance* will have *more charge*.

DISCUSSION PROBLEM [24.1]:

The Earth is a conductor of radius 6400km. If it were an isolated sphere what would be its capacitance?

Capacitance of two parallel plates:



We can use a battery or a generator to move charge Q from one plate to the other. The electric field between the plates is:

$$E = \frac{\sigma}{\varepsilon_\circ} = \frac{Q}{\varepsilon_\circ A}. \qquad \text{(From ch. 22)}$$

Also, the potential difference (*voltage*) between the two plates is:

$$V = E.d = \frac{\sigma}{\varepsilon_\circ} d. \qquad \text{(From ch. 23)}$$

So the capacitance of this pair of plates is

$$C = \frac{Q}{V} = \frac{\sigma A}{V} = \varepsilon_\circ \frac{A}{d}.$$

Two parallel plates (continued):

Practical considerations

***Example*:** $A = 5.0cm \times 5.0cm$

$d = 0.5cm.$

$$\therefore C = \varepsilon_\circ \frac{A}{d} = 4.4 \times 10^{-12} F = 4.4pF.$$

Maximum possible value of E in air (from earlier) $\approx 3 \times 10^6$ V/m. Therefore, the maximum potential difference (voltage) we can get between this pair of plates is:

$$V_{max} = E_{max}.d \approx 3 \times 10^6 \times 0.005 \approx 15{,}000V$$

Note: *it depends only on the spacing*.

Also,

$$Q_{max} = CV_{max} = 4.4 \times 10^{-12} \times 15 \times 10^3 = 66nC$$

$$\therefore U_{max} = \frac{1}{2} Q_{max} V_{max} = 4.95 \times 10^{-4} J.$$

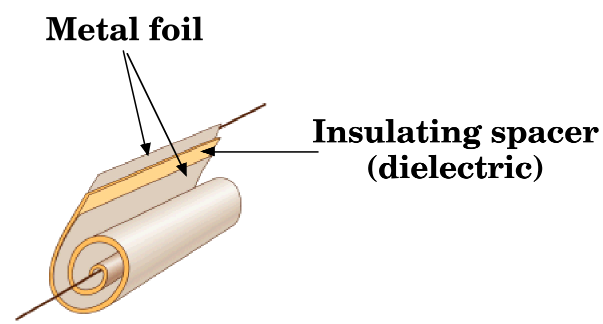
To have a 1F capacitance the area would have to be $A \approx 5.6 \times 10^8 m^2$, i.e., the length of each side of the plates would be $\approx$ 23.8km (i.e., about 14 miles!) with a spacing of 0.5cm.

Two parallel plates:

$$C = \varepsilon_\circ \frac{A}{d}$$

How can we increase the capacitance, i.e, get more charge per unit of potential difference?

- ***increase* A**
- ***decrease* d**

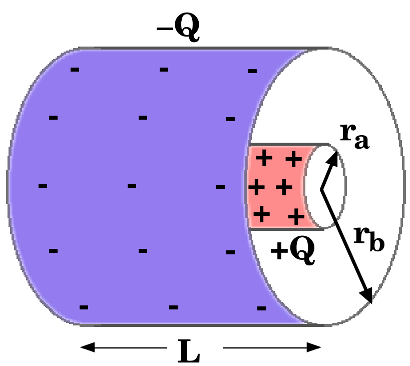


The capacitor is rolled up into a cylindrical shape.

- ***increase*** $\varepsilon_\circ$

by changing the medium between the plates, i.e., $\varepsilon_\circ \Rightarrow \varepsilon = \kappa\varepsilon_\circ$ **(later).**

A cylindrical (coaxial) capacitor:

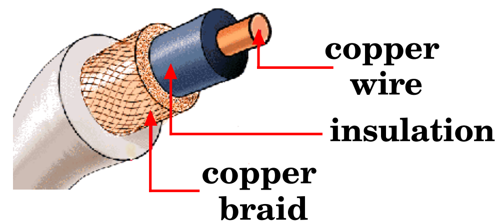


The capacitance of a coaxial capacitor of length L is:

$$C = \frac{2\pi\varepsilon_\circ L}{\ln(r_b/r_a)}$$

A cylindrical (coaxial) capacitor (continued):

Example: coaxial (antenna) wire.



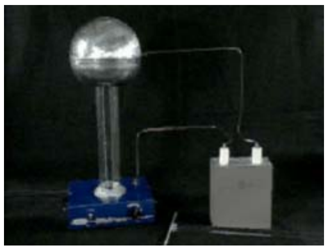
Assume an outer conductor (braid) radius $r_b \approx 2.5$mm, and an inner conductor (wire) radius $r_a \approx 0.5$mm, with neoprene insulation ($\varepsilon_\circ \Rightarrow \kappa\varepsilon_\circ = 6.9\varepsilon_\circ$).

The capacitance per meter is:

$$\frac{C}{L} = \frac{2\pi\kappa\varepsilon_\circ}{\ln(r_b/r_a)}$$

$$= \frac{2 \times \pi \times 6.9 \times 8.85 \times 10^{-12}}{1.609} = 2.384 \times 10^{-10}\ \text{F/m}$$

$$= 238.4\ \text{pF/m}$$



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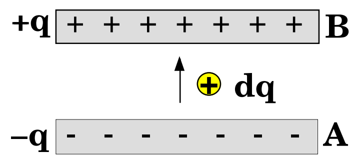
Because work is done to charge a capacitor, the capacitor stores energy, *electrostatic potential energy*. The energy is released when the capacitor is discharged.

Where is the energy stored?

... in the electric field (between the plates)!

Storing energy in a capacitor:



To store energy in a capacitor we "charge" it, producing an electric field. We do work moving charges from plate A to plate B. If the plates already have charge ±q and dq is then moved from A to B, the work done is

$$dW = -dq(v_A - v_B),$$

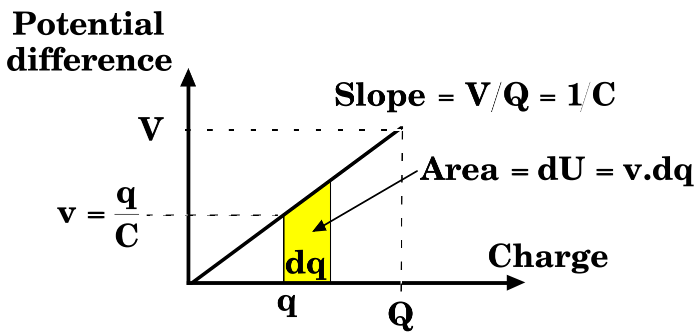
where v_A and v_B are the potentials of plates A and B, respectively. This, then is the incremental *increase* in potential energy, dU, of the capacitor system. If $v = v_B - v_A$ ($v_B > v_A$) then $dU = vdq$.

$$\text{But} \qquad v = \frac{q}{C}, \qquad \text{(by definition)}$$

so the incremental increase in energy when dq of charge is taken from A → B is:

$$\therefore dU = \left(\frac{q}{C}\right)dq.$$

Energy stored in a capacitor (continued)



So, in charging a capacitor from $0 \rightarrow Q$ the *total* increase in potential energy is:

$$U = \int dU = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2C}\left[q^2\right]_0^Q = \frac{1}{2}\frac{Q^2}{C}$$

$$\left(= \frac{1}{2}QV = \frac{1}{2}CV^2\right)$$

i.e., the area under the V – Q plot. This potential energy can be recovered when the capacitor is *discharged*, i.e., when the stored charge is released.

(NOTE: this is the same we obtained earlier for a charged conducting sphere.)

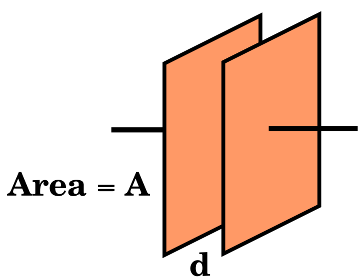
Energy density of an electric field ...

Assume we have a parallel plate capacitor, then stored energy is:

$$U = \frac{1}{2}CV^2.$$

But $C = \varepsilon_\circ \frac{A}{d}$ and $V = E.d$,

$$\therefore U = \frac{1}{2}\varepsilon_\circ \frac{A}{d}(E.d)^2 = \frac{1}{2}\varepsilon_\circ (A.d)E^2.$$

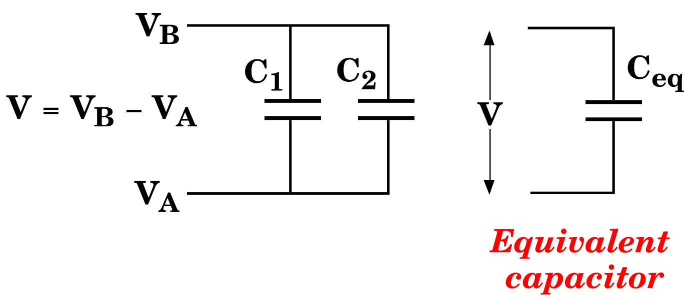


But $A.d \Rightarrow$ volume of the electric field. So the *energy density* is:

$$u_e = U/\text{volume} = \frac{1}{2}\varepsilon_\circ E^2.$$

This result is true for <u>*all*</u> electric fields.

Combining capacitors (parallel):



The charges on the capacitors are:

$$Q_1 = C_1 V \quad \text{and} \quad Q_2 = C_2 V$$

and the total charge stored is:

$$Q = Q_1 + Q_2 = (C_1 + C_2)V = C_{eq}V.$$

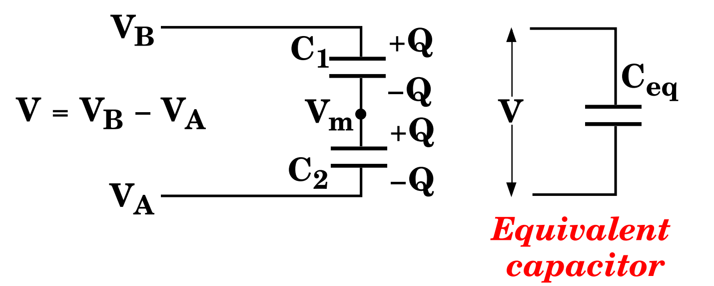
So, this combination is equivalent to a single capacitor with capacitance

$$C_{eq} = C_1 + C_2.$$

When more than two capacitors are connected in parallel:

$$C_{eq} = \sum_i C_i.$$

Combining capacitors (series):



The charge on the capacitors is the same: $\pm Q$.

The individual potential differences are:

$$V_1 = (V_B - V_m) = \frac{Q}{C_1} \quad \text{and} \quad V_2 = (V_m - V_A) = \frac{Q}{C_2}.$$

Therefore, the total potential difference is:

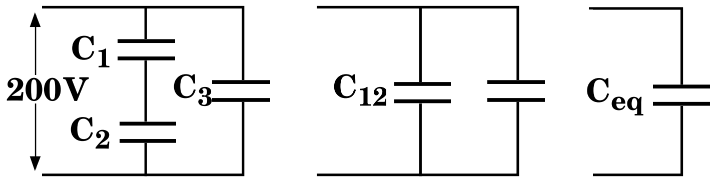
$$\therefore V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_{eq}},$$

providing $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$.

With more than two capacitors:

$$\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i}$$

Problem 24.39 page 779 :



(a) C_1 and C_2 are in series:

$$\therefore \frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{4\mu F} + \frac{1}{15\mu F} = 0.3167 \times 10^6$$

$$\therefore C_{12} = 3.16\mu F$$

C_{12} and C_3 are in parallel:

$$\therefore C_{eq} = C_{12} + C_3 = 3.16\mu F + 12\mu F = 15.16\mu F.$$

(b) We have: $Q_1 = Q_2 \quad (= Q)$

and $V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$

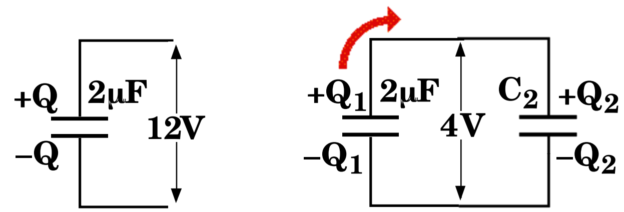
$$\therefore 200 = 0.3167 \times 10^6 Q,$$

i.e., $Q = 0.63 \times 10^{-3}$C

$$Q_3 = C_3 V = 12 \times 10^{-6} \times 200 = 2.4 \times 10^{-3} C$$

(c) $U = \frac{1}{2} C_{eq} V^2 = \frac{1}{2} \times 15.16 \times 10^{-6} \times 200^2 = 0.30J$

Problem 24.58 page 781:



Initially, the charge on the 2μF capacitor is

$$Q = C_1 V = 2 \times 10^{-6} \times 12 = 24 \times 10^{-6} C.$$

When connected across the second capacitor, this charge is redistributed (none is lost!!). The "new" charges are

$$Q_1 = C_1 V' = 2 \times 10^{-6} \times 4 = 8 \times 10^{-6} C.$$

$$Q_2 = C_2 V' = C_2 \times 4 = 4C_2.$$

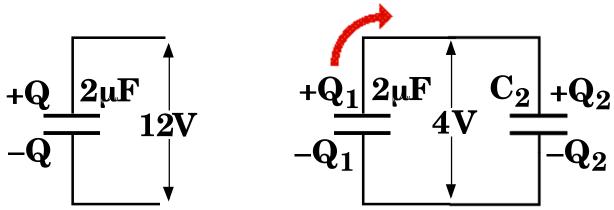
But total charge is conserved ... *where could it go*? So, $Q_1 + Q_2 = Q$. $\therefore Q_2 = Q - Q_1$.

i.e., $4C_2 = (24 \times 10^{-6}) - (8 \times 10^{-6}) = 16 \times 10^{-6}$.

$$\therefore C_2 = 4 \times 10^{-6} F = 4\mu F.$$

What about stored energy *before* and *after*?

Problem 24.58 page 781: continued ...



What about energy *before* and *after*?

- **Energy *before*** $\Rightarrow \frac{1}{2}C_1V^2$

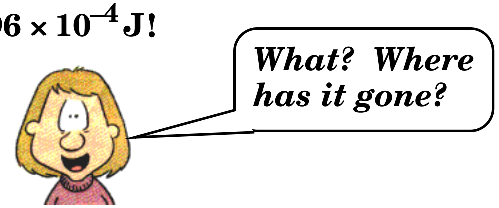
$$= \frac{1}{2} \times 2 \times 10^{-6} \times 12^2 = \boxed{1.44 \times 10^{-4}\,\mathrm{J}}$$

- **Energy *after*** $\Rightarrow \frac{1}{2}C_1V'^2 + \frac{1}{2}C_2V'^2$

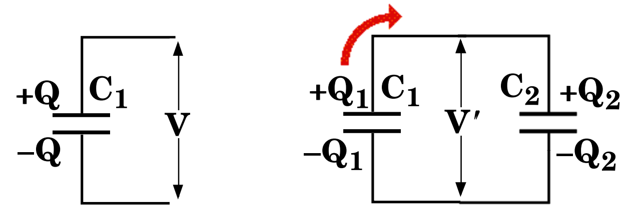
$$= \frac{1}{2} \times 2 \times 10^{-6} \times 4^2 + \frac{1}{2} \times 4 \times 10^{-6} \times 4^2$$

$$= \boxed{0.48 \times 10^{-4}\,\mathrm{J}}$$

i.e, a *loss* of 0.96×10^{-4} J!



Further analysis ...



If C_2 is uncharged initially ...

- **Initial energy** $U_i = \frac{1}{2}QV.$
- **Final energy** $U_f = \frac{1}{2}Q_1V' + \frac{1}{2}Q_2V'$

$$= \frac{1}{2}(Q_1 + Q_2)V' = \frac{1}{2}QV'.$$

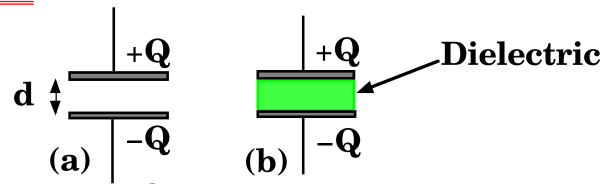
$$\therefore \frac{U_f}{U_i} = \frac{V'}{V}.$$

But $Q_1 + Q_2 = C_1V' + C_2V' = (C_1 + C_2)V'.$

Also $Q_1 + Q_2 = Q = C_1V$ **... i.e.,** $\frac{V'}{V} = \frac{C_1}{C_1 + C_2}.$

$$\therefore \frac{U_f}{U_i} = \frac{C_1}{C_1 + C_2} \quad \text{always} < 1.$$

Dielectrics:



$$C_\circ = \frac{Q}{V_\circ} \qquad C = \frac{Q}{V}$$

(a) The electric field in an isolated charged parallel plate capacitor (in vacuum) is: $E_\circ = \dfrac{\sigma}{\varepsilon_\circ}$.

$$\therefore V_\circ = E_\circ d = \frac{\sigma}{\varepsilon_\circ} d.$$

(b) When a dielectric is inserted, $\varepsilon_\circ \Rightarrow \kappa\varepsilon_\circ$, where κ is the *dielectric constant*, so $E = \dfrac{\sigma}{\kappa\varepsilon_\circ}$.

$$\therefore V = E.d = \frac{\sigma}{\kappa\varepsilon_\circ} d = \frac{V_\circ}{\kappa},$$

i.e., the potential difference is *reduced*.

$$\therefore C = \frac{Q}{V} = \frac{\kappa Q}{V_\circ} = \kappa C_\circ \qquad \left(= \kappa\varepsilon_\circ \frac{A}{d}\right)$$

i.e., *the capacitance increases by* κ.

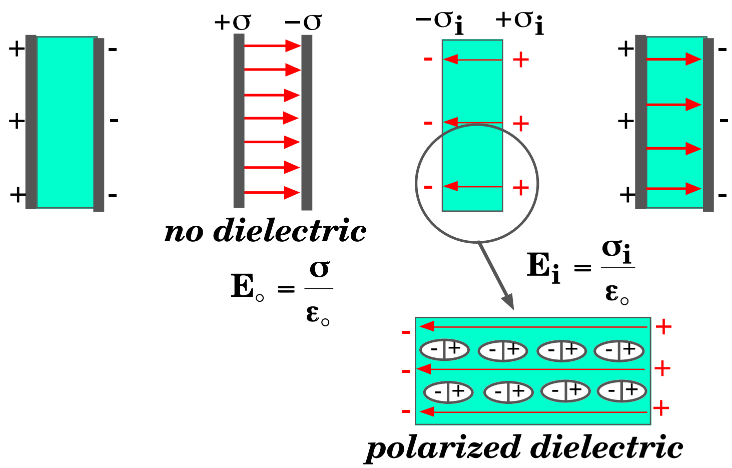
Dielectrics (continued):

Three main advantages:

- **maintains plate separation when small,**
- **increased capacitance for a given size.**
- **dielectric increases the max. electric field possible (*dielectric strength*), and hence potential difference, between the plates before breakdown, $V_{max} = E_{max}.d$.**

Material	κ	Dielectric Strength (E_{max})
Air	1.00059	3×10^6 V/m
Paper	3.7	16×10^6 V/m
Neoprene	6.9	12×10^6 V/m
Polystyrene	2.55	24×10^6 V/m

How does a dielectric work? ...

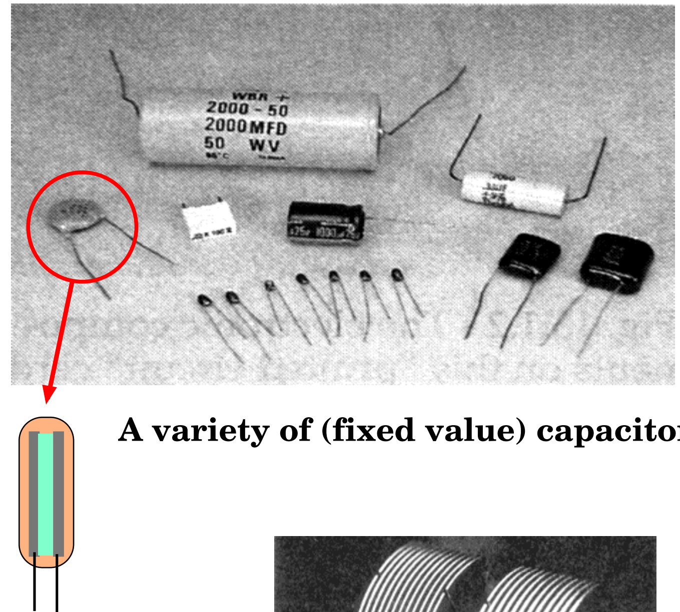


Hence the electric field in the presence of a dielectric is *reduced* to:

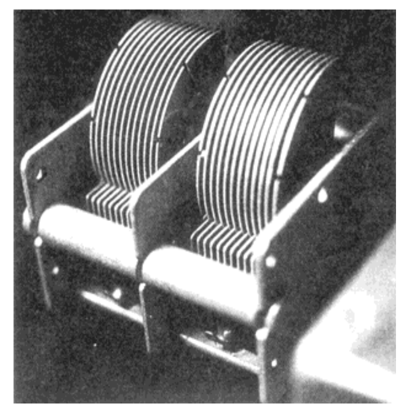
$$E = E_\circ - E_i = \frac{\sigma}{\varepsilon_\circ} - \frac{\sigma_i}{\varepsilon_\circ} = \frac{\sigma - \sigma_i}{\varepsilon_\circ}.$$

Therefore, the potential difference V (= E.d) is reduced also. Consequently, $C > C_\circ$.

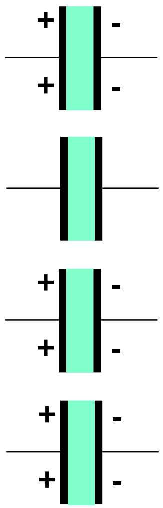
** ***Show that the induced charge***: $\sigma_i = \frac{(\kappa - 1)}{\kappa}\sigma.$



A variety of (fixed value) capacitors



A simple variable (air) capacitor



“*charged*” 1

“*uncharged*” 0

“*charged*” 1

“*charged*” 1

Binary [1 0 1 1] ⇒ Decimal 11

Dynamic random access memory (DRAM) is composed of banks of capacitors. A “charged” capacitor represents the binary digit “1” and “uncharged” capacitor represents the binary digit “0”.

A 32MB DRAM chip contains 256 million capacitors!

Conceptual problem [1]:

(a) **(b)**

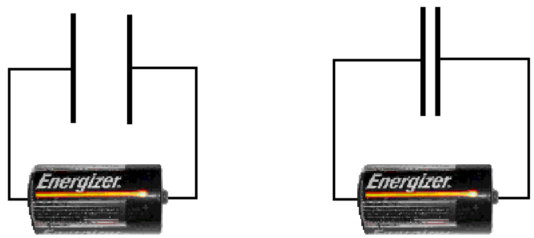
A parallel plate capacitor is charged by a generator. The generator is then disconnected (a). If the spacing between the plate is decreased (b), what happens to:

[1] the charge on the plates
[2] the potential across the plates
[3] the energy stored by the capacitor.

The choices in each case are:

A: the same
B: it increases
C: it decreases

Conceptual problem [2]:



(a) (b)

A parallel plate capacitor is charged by a generator (a). When fully charged, and while the generator is still connected, the spacing between the plate is decreased (b), what happens to:

[1] the potential across the plates
[2] the charge on the plates
[3] the energy stored by the capacitor.

The choices in each case are:

A: the same
B: it increases
C: it decreases