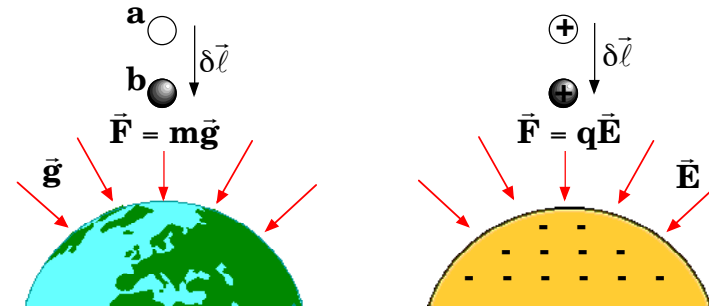


## CHAPTER 23

### ELECTRIC POTENTIAL

- Potential difference and electric field
- Potential difference between two parallel plates
- Potential due to a single point charge
- Potential due to a collection of charges
  - † *Work done bringing charges together*
- Potential for continuous charge distributions
  - † *Charged, hollow sphere*
  - † *Uniformly charged ring*
- **Equipotential surfaces (You study using the handout together with section 23-5 in the textbook.)**

*Work done and potential energy ...*



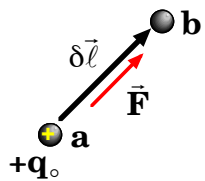
Let's look at the similarity between electric and gravitational fields. The work done *by the g-field* in moving the mass from a  $\rightarrow$  b is:

$$\delta W = \int_a^b \vec{F} \cdot d\vec{\ell} = mg\delta\ell \quad (> 0), \text{ i.e., positive work.}$$

The *change in potential energy* of the mass in the gravitational field in moving from a  $\rightarrow$  b is:

$$\delta U = U_b - U_a = -mg\delta\ell \quad (< 0), \text{ i.e., a loss.}$$
$$\therefore \delta W = -\delta U = -(U_b - U_a).$$

Note: the work done by the g-field ( $\delta W$ ) in moving the mass from a  $\rightarrow$  b is the same as the work *you do* in raising the mass from b  $\rightarrow$  a.



If a charge moves through a displacement  $\delta \vec{l}$  in a field  $\vec{E}$ , the work done **by the E-field** in moving the charge from  $a \rightarrow b$  is

$$\delta W = \vec{F} \cdot \delta \vec{l} = -\delta U,$$

where  $\delta U$  is the change in **potential energy** of the charge in the  $\vec{E}$ -field. But  $\vec{F} = q_0 \vec{E}$ , so the change in potential energy is:

$$\delta U = (U_b - U_a) = -\vec{F} \cdot \delta \vec{l} = -q_0 \vec{E} \cdot \delta \vec{l}.$$

We define the **electric potential** (V) as the potential energy per unit charge,

$$\text{i.e., } V = \frac{U}{q_0}.$$

So the **potential difference** between b and a

$$\delta V = V_b - V_a = V_{ba}$$

is given by:

$$\delta V = \frac{\delta U}{q_0} = -\vec{E} \cdot \delta \vec{l}.$$

We will work problems in *Cartesian* (x,y,z) and *polar* (r,θ,φ) coordinate systems.

[1] *Cartesian coordinates*:

$$\vec{E} \Rightarrow (E_x, E_y, E_z) \text{ and } \delta \vec{l} \Rightarrow (\delta x, \delta y, \delta z)$$

$$\delta V = -\vec{E} \cdot \delta \vec{l} = -(E_x \hat{i} + E_y \hat{j} + E_z \hat{k}) \cdot (\delta x \hat{i} + \delta y \hat{j} + \delta z \hat{k})$$

$$\text{i.e., } \delta V = -(E_x \delta x + E_y \delta y + E_z \delta z).$$

$$\therefore E_x = -\frac{dV}{dx}, E_y = -\frac{dV}{dy}, E_z = -\frac{dV}{dz}$$

$$\text{i.e., } \vec{E} = -\left(\frac{dV}{dx} \hat{i} + \frac{dV}{dy} \hat{j} + \frac{dV}{dz} \hat{k}\right).$$

These are the basic relationships between the electric field,  $\vec{E}$ , and the electric potential, V, in Cartesian coordinates.

**UNITS:**

If  $E \Rightarrow \text{N/C}$  and  $r \Rightarrow \text{m}$   
then:  $V \Rightarrow \text{volts (V)}$

But, by definition:

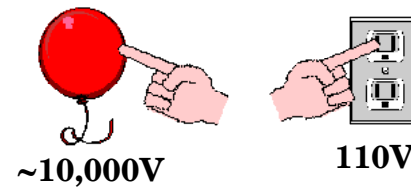
$$E_x = -\frac{dV}{dx}, \text{ etc.,}$$

then:  $E \Rightarrow \text{V/m,}$

which means that  $\text{N/C} \equiv \text{V/m.}$

(It is more usual to use  $\text{V/m}$  as the unit of electric field.)

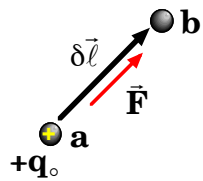
**DISCUSSION PROBLEM [23.1]:**



When you charge a balloon by friction, its electric potential is  $\sim 10,000\text{V}$ , but it is safe to handle! And yet, a typical socket operates at a potential of  $110\text{V}$  but will give you a (potentially!) fatal shock.

- \* What's the difference?
- \* Why is the socket more "shocking"?

**Conventional definition of work done in an electric field ...**



The work done **by the field** in moving the charge from  $a \rightarrow b$  is

$$\begin{aligned} \delta W &= -\delta U = -(U_b - U_a) \\ &= -(q_0 V_b - q_0 V_a) = q_0 (V_a - V_b). \end{aligned}$$

(Remember, by definition  $\Rightarrow V = U/q_0$ )

Conventionally, when a charge moves from  $a \rightarrow b$  we write **the work done by the field** as:

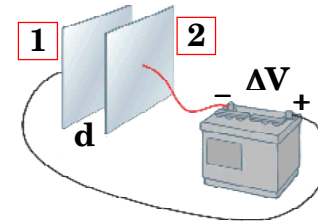
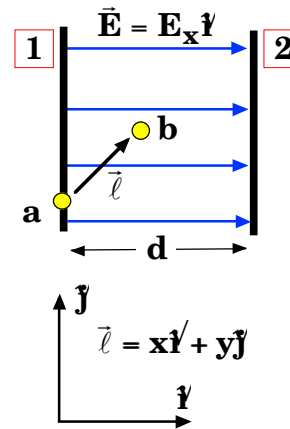
$$\delta W = q_0 (V_a - V_b) = q_0 V_{ab},$$

where  $V_{ab}$  is the **potential difference** between the **start point** (a) and the **end point** (b). (Note also if the charge was released and free to move in the field,  $\delta K = \delta W$ .)

$\therefore$  **The work done by you** in moving a charge from  $a \rightarrow b$  is:

$$\delta W = -q_0 V_{ab} = -q_0 (V_a - V_b).$$

**Example using Cartesian coordinates ...  
Potential between two parallel charges plates**



Find the potential difference between a and b (a displacement  $\vec{l}$ ) in a field  $\vec{E} = E_x \hat{i}$  produced between

two parallel, infinitely large charged plates, spaced a distance  $d$  apart. Along the displacement, the change in potential is:

$$dV = -\vec{E} \cdot d\vec{l} = -E_x \hat{i} \cdot (dx \hat{i} + dy \hat{j}) = -E_x dx.$$

$$\therefore V_b - V_a = \int_a^b dV = -E_x \int_a^b dx = -E_x x = -\frac{\sigma}{\epsilon_0} x$$

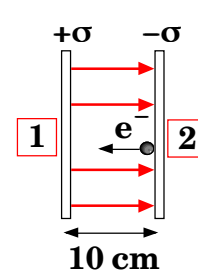
$$\therefore V_b = V_a - \frac{\sigma}{\epsilon_0} x$$

$V_b = V_a - \frac{\sigma}{\epsilon_0} x$   
 $\therefore (V_2 - V_1) = \Delta V = -\frac{\sigma}{\epsilon_0} d,$   
 i.e.,  $V_1 > V_2$ .

- $\Delta V$  *is independent of y*, it depends only on  $\sigma$  and  $d$ . Thus,  $\Delta V$  is the same between *any* point on plate **1** and *any* point on plate **2**. This means that the potential is constant over an infinitely charged plate.
- The work done *by the field* in moving a charge  $q$  from  $a \rightarrow b$  is:  

$$\delta W = q(V_1 - V_2) > 0,$$
 so a +ve charge moves from a position of higher potential ( $V_1$ ) to lower potential ( $V_2$ ).

**Problem 23.22, page 743:**



(a) Since the field would move a +ve charge from the  $+\sigma$  plate to the  $-\sigma$  plate, the  $+\sigma$  plate is at the higher potential.

$$\Delta V = (V_1 - V_2) = 500\text{V} = -E \cdot d,$$

$$\therefore |E| = \Delta V/d = 500/0.1 = 5000 \text{ V/m}.$$

(b) Work done *by the field* is  $W_{2 \rightarrow 1} = qV_{21}$ ,

$$\text{But } V_{21} = (V_2 - V_1) = -500 \text{ V}.$$

$$\therefore W_{2 \rightarrow 1} = -1.6 \times 10^{-19} \times (-500) = 8.0 \times 10^{-17} \text{ J}.$$

(c) The change in potential energy of the electron:  $\Delta U = U_1 - U_2 = qV_1 - qV_2 = q(V_1 - V_2)$

$$= -1.6 \times 10^{-19} \times 500 = -8.0 \times 10^{-17} \text{ J}.$$

Mechanical energy is conserved:

$$\therefore \Delta K = -\Delta U = 8.0 \times 10^{-17} \text{ J}.$$

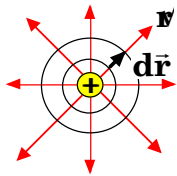
$$\Delta K = \frac{1}{2} m(v^2 - \cancel{v_0^2}). \quad \therefore v = \sqrt{\frac{2\Delta K}{m}} = 1.33 \times 10^7 \text{ m/s}.$$

$$\text{Note also: } \Delta K = W_{2 \rightarrow 1} = qV_{21}.$$

(Work - energy theorem).

[2] *Polar coordinates*

If the electric field has radial symmetry, i.e., it depends only on  $|\vec{r}|$ , e.g., a point charge, then



$$\vec{E}(\vec{r}) \Rightarrow E_r \hat{r} = k \frac{Q}{r^2} \hat{r}.$$

For a radial displacement  $d\vec{r}$  (in the  $\hat{r}$  direction):

$$dV(\vec{r}) = -\vec{E}(\vec{r}) \cdot d\vec{r} = -E_r \hat{r} \cdot d\vec{r}.$$

But  $d\vec{r} \parallel \hat{r}$ ,

$$\therefore \hat{r} \cdot d\vec{r} = |\hat{r}| |d\vec{r}| \cos 0 = dr.$$

$\therefore$  potential difference between radii  $r_2$  and  $r_1$ :

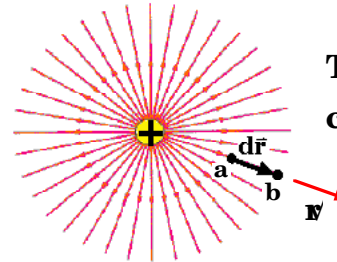
$$V_{21} = V(r_2) - V(r_1) = - \int_{r_1}^{r_2} E_r dr.$$

But  $dV(\vec{r}) = -E_r dr$ , so the radial electric field is

$$E_r = - \frac{dV(\vec{r})}{dr}.$$

Again, we have simple relations between the electric field  $E_r$  and the electric potential  $V(\vec{r})$ .

*Example ... electric potential for a point charge:*



The electric field of a point charge is:

$$\vec{E}(\vec{r}) = k \frac{Q}{r^2} \hat{r}.$$

For a small displacement  $d\vec{r}$  in the radial direction ( $\hat{r}$ ), the change in potential is:

$$dV(\vec{r}) = -\vec{E}(\vec{r}) \cdot d\vec{r} = -k \frac{Q}{r^2} \hat{r} \cdot d\vec{r} = -k \frac{Q}{r^2} dr.$$

$$\therefore V(\vec{r}) = -k \int \frac{Q}{r^2} dr = k \frac{Q}{r} + V_0,$$

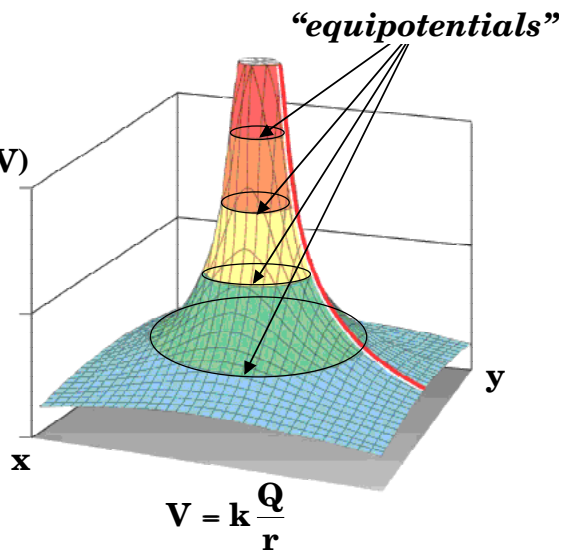
where  $V_0$  is an integration constant. If we define the electric potential at infinity as zero, i.e.,  $V(\vec{r} \rightarrow \infty) = 0$ , then  $V_0 = 0$ . So,

$$V(\vec{r}) = k \frac{Q}{r},$$

is the *absolute* electric potential at the point  $\vec{r}$ .

$$\therefore V_b - V_a = kQ \left[ \frac{1}{r_b} - \frac{1}{r_a} \right] \quad (< 0 \text{ if } Q \text{ is } +ve)$$

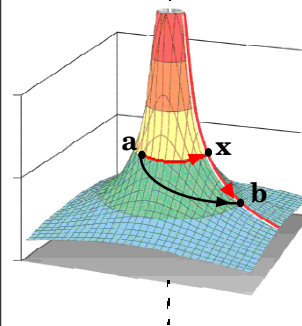
Electric potential (V)



The electric potential for a positive charge. If the charge is negative, the potential looks like a “hole” rather than a “hill”.

Note that as  $x$  (and  $y$ )  $\rightarrow \pm\infty$ ,  $V \rightarrow 0$ .

Go from  $a \rightarrow b$  by different routes. The



potential at any point a distance  $r$  from a point charge is:

$$V(r) = k \frac{Q}{r}$$

Since  $r_a = r_x$ ,  $V_a = V_x$

so, the **potential difference**

$$V_{ab} = V_{xb},$$

i.e., the potential difference between two points **does not depend on the path between them** only the potentials at the end points.

The work done by you in moving a charge  $q$  from  $a \rightarrow b$  by the two different routes is:

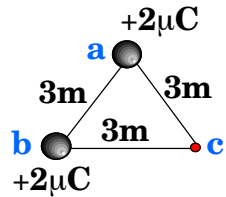
$$[1] \quad W_{a \rightarrow b} = -q(V_a - V_b) = -qV_{ab}.$$

$$[2] \quad W_{a \rightarrow x \rightarrow b} = [-q(V_a - V_x)] + [-q(V_x - V_b)] \\ = -q(V_x - V_b) = -qV_{xb}.$$

$$\text{But } V_{ab} = V_{xb} \quad \therefore W_{a \rightarrow b} = W_{a \rightarrow x \rightarrow b}.$$

So, the work done by you ( $= -q\Delta V$ ) in moving a charge from one point to another **does not depend on the path** ... only on  $\Delta V$ .

**Problem 23.30, page 743:**



$$V = k \frac{Q}{r}$$

(a) Potential at **c** is due to both  $Q_a$  and  $Q_b$

$$V_c = k \frac{(+2 \times 10^{-6})}{3} + k \frac{(+2 \times 10^{-6})}{3}$$

$$= 1.2 \times 10^4 \text{ V}$$

(b) Work done by you in bringing a charge  $q_0$  from  $\infty$  to the point **c** is:

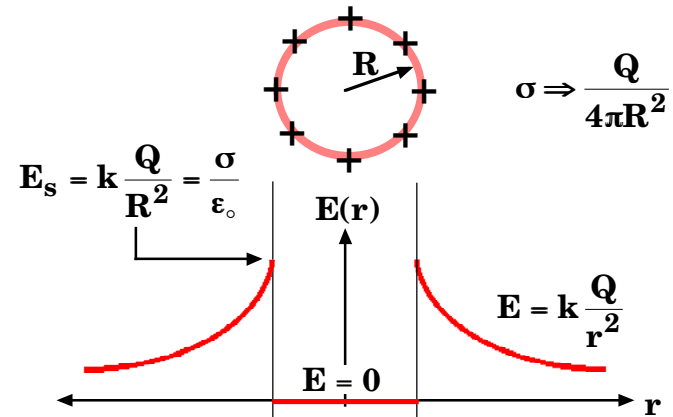
$$W = -q_0 \Delta V = -q_0 (V_\infty - V_c) = q_0 V_c$$

$$= 5 \times 10^{-6} \times 1.2 \times 10^4$$

$$= 6 \times 10^{-2} \text{ J}$$

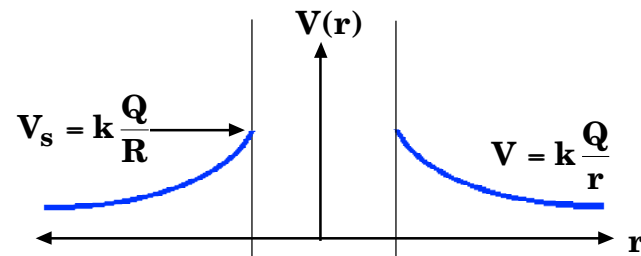
(c) if  $Q_b = -2\mu\text{C}$  then  $V_c = 0$  and  $W = 0$ .

**Potential due to a spherical shell of charge (on a hollow or solid conducting sphere) ...**



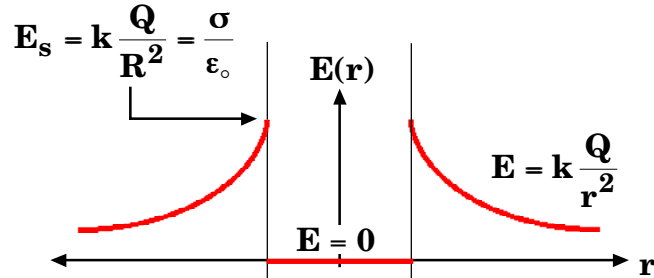
From earlier:  $E = -\frac{dV}{dr}$  so  $dV = -E \cdot dr$ ,

$$\therefore V(r > R) = - \int_{r>R} E \cdot dr = -k \int \frac{Q}{r^2} dr = k \frac{Q}{r}$$



*But what about inside the sphere?*

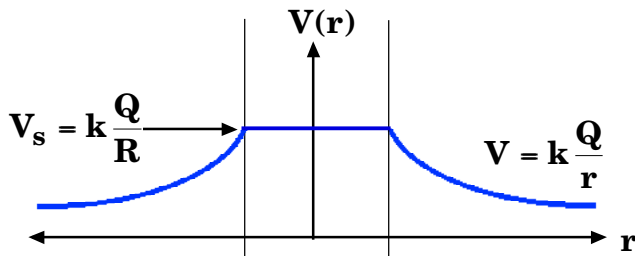
**Potential due to a spherical shell of charge (on a hollow or solid conducting sphere) ...**



Inside the sphere, i.e., for  $r < R$ ,  $dV = -E \cdot dr = 0$   
 $\therefore V(r < R) = \text{constant}$ .

If  $V$  is constant inside sphere, no work is done in moving a charge *anywhere* inside the sphere. Then  $W = -q_0 \Delta V = 0$ , i.e.,  $\Delta V = 0$ .

$$\therefore V(r < R) = k \frac{Q}{R} \Rightarrow \text{constant}$$



**TWO POINTS:**

- [1]  $V$  is constant inside a conducting sphere (i.e., the same as at the surface).
- [2] At the surface:  $V_s = k \frac{Q}{R}$  and  $E_s = k \frac{Q}{R^2}$ .

$$\therefore V_s = E_s \cdot R \text{ or } E_s = \frac{V_s}{R}.$$

As the charge  $Q$  on the sphere increases, so do

$$V_s \left( = k \frac{Q}{R} \right) \text{ and } E_s \left( = k \frac{Q}{R^2} \right).$$

Under “normal conditions” *the max electric field obtainable in air* before breakdown is

$$E_{\text{max}} \sim 3 \times 10^6 \text{ V/m.}$$

This sets a maximum potential and a maximum charge for a spherical conductor (radius  $R$ ):

$$\text{i.e., } V_{\text{max}} = E_{\text{max}} \cdot R \sim 3 \times 10^6 R \text{ volts.}$$

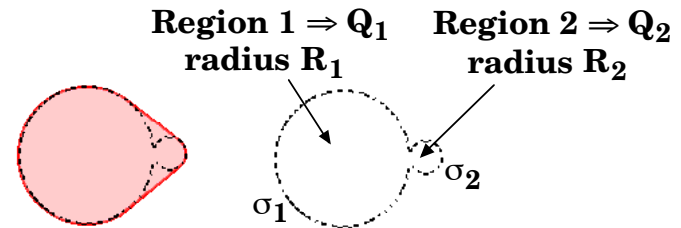
$$\text{Since, } E = k \frac{Q}{R^2}: Q_{\text{max}} \sim \frac{3 \times 10^6}{k} R^2 \text{ Coulombs.}$$

Larger  $R$  means larger  $V_{\text{max}}$  and  $Q_{\text{max}}$  before breakdown.

**DISCUSSION PROBLEM [23.2]**

If the electric potential of the Earth is very large, how come we aren't *fried* to a crisp when standing barefoot on the Earth's surface?

*Also, we can now show why charges “pile-up” at sharp points on a charged conductor ...*



The potential *inside* the conductor is constant

$$\therefore k \frac{Q_1}{R_1} = k \frac{Q_2}{R_2}.$$

But  $Q_1 \approx 4\pi R_1^2 \sigma_1$  and  $Q_2 \approx 4\pi R_2^2 \sigma_2$

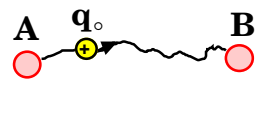
so  $R_1 \sigma_1 \approx R_2 \sigma_2$

i.e.,  $\sigma_2 \approx \frac{R_1}{R_2} \sigma_1$ .  $\therefore \sigma_2 > \sigma_1$ .

Also,  $E_{1s} = \frac{\sigma_1}{\epsilon_0}$  and  $E_{2s} = \frac{\sigma_2}{\epsilon_0}$ .  $\therefore E_{2s} > E_{1s}$ .

Therefore, the charge density and the surface electric field are greater at “points”.

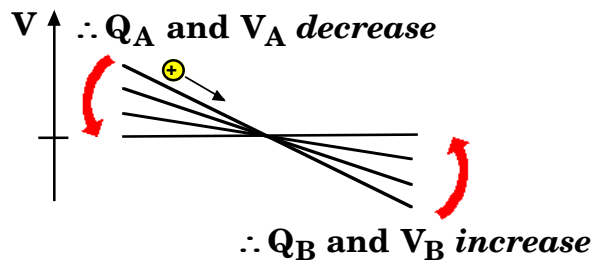
Take two charged spheres A ( $V_A$ ) and B ( $V_B$ ) connected by a conducting wire and  $V_A > V_B$ .

$$V_A = k \frac{Q_A}{R_A} \quad \text{A} \quad q_0 \quad \text{B} \quad V_B = k \frac{Q_B}{R_B}$$


The force on the charge  $q_0$  is:  $F = q_0 E = -q_0 \frac{dV}{dl}$ ,

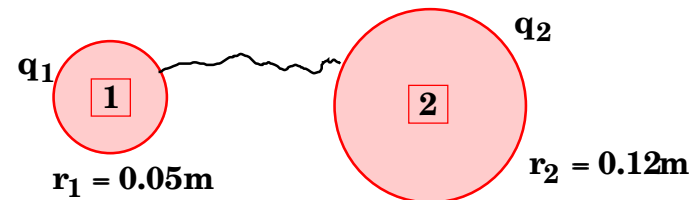
but  $V_A > V_B \dots \therefore \frac{dV}{dl} < 0$ , so +ve charges move

from A to B, i.e., from **high potential** ( $V_A$ ) to **low potential** ( $V_B$ ).



As charges move from A to B,  $V_A$  decreases and  $V_B$  increases. When  $V_A = V_B$ , charges stop moving, because when  $\Delta V = 0$  then  $F = 0$ .

**Problem 23.64, page 745:**



We have:  $V_1 = k \frac{q_1}{r_1} = E_{1s} r_1$ .

$$V_2 = k \frac{q_2}{r_2} = E_{2s} r_2.$$

But  $V_1 = V_2$ , so  $E_{1s} r_1 = E_{2s} r_2$ .

$$\therefore E_{1s} = \frac{r_2}{r_1} E_{2s} = \frac{0.12}{0.05} \times 200 = 480 \text{ kV/m}.$$

From earlier,  $E_{1s} = \frac{\sigma_1}{\epsilon_0}$  and  $E_{2s} = \frac{\sigma_2}{\epsilon_0}$ .

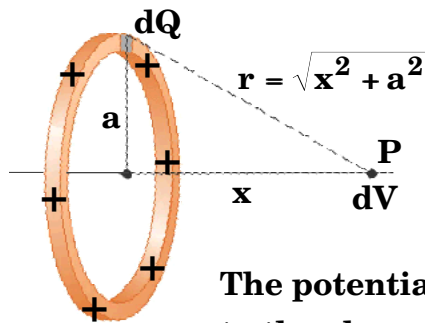
$$\therefore \sigma_1 = \epsilon_0 E_{1s} = 8.85 \times 10^{-12} \times 480 \times 10^3$$

$$= 4.25 \times 10^{-6} \text{ C/m}^2$$

and  $\sigma_2 = \epsilon_0 E_{2s} = 8.85 \times 10^{-12} \times 200 \times 10^3$

$$= 1.77 \times 10^{-6} \text{ C/m}^2$$

**Uniformly charged ring:**



The potential at the point P due to the element of charge dQ is:

$$dV = k \frac{dQ}{r}$$

$$\therefore V(x) = \int k \frac{dQ}{r},$$

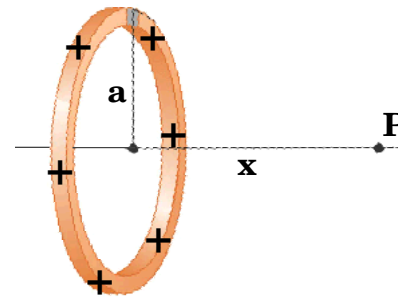
but  $r = \sqrt{x^2 + a^2}$ , which is constant.

$$\therefore V(x) = \frac{k}{\sqrt{x^2 + a^2}} \int dQ$$

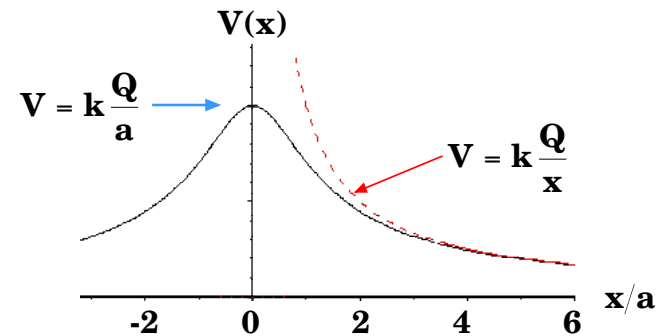
$$= k \frac{Q}{\sqrt{x^2 + a^2}}$$

where Q is the total charge on the ring.

**Uniformly charged ring:**



$$V(x) = k \frac{Q}{\sqrt{x^2 + a^2}}$$



When  $x = 0$ ,  $V(x) = k \frac{Q}{a}$

When  $x \gg a$ ,  $V(x) \Rightarrow k \frac{Q}{x}$ ,

i.e., it looks like a point charge.