The Vector (or cross) product

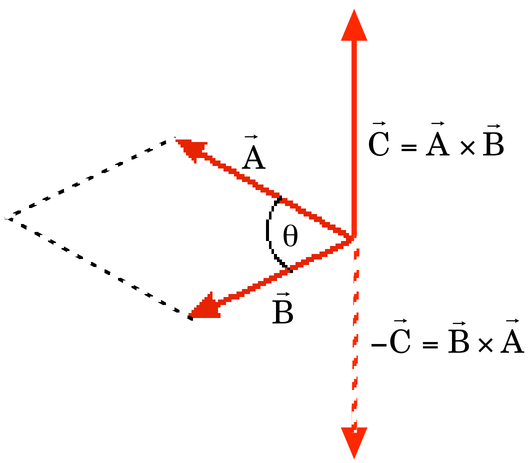
Given two vectors $\vec{A}$ and $\vec{B}$ with an angle θ between them; the **vector product** (or **cross product**) is defined as:

$$\vec{A} \times \vec{B} = \vec{C},$$

where $\vec{C}$ is a vector whose magnitude is given by:

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta,$$

and whose direction is given by the **right-hand-rule**.



To find the direction of $\vec{C}$, clench the fingers of the **right hand** as shown. If the fingers curl in the direction *from* the $\vec{A}$ vector *to* the $\vec{B}$ vector, then direction of vector $\vec{C}$ is given by the thumb. Note, the vector $\vec{C}$ is perpendicular to the plane containing the $\vec{A}$ and $\vec{B}$ vectors. There are several important points:

[1] The order of the vectors is important. Note that $\vec{A} \times \vec{B}$ is **not** the same as $\vec{B} \times \vec{A}$. In fact, satisfy yourself that

$$(\vec{A} \times \vec{B}) = -(\vec{B} \times \vec{A}),$$

i.e., the vector product is *not commutative*.

[2] The vector product obeys the *distributive* rule; i.e., given any three vectors, $\vec{A}$, $\vec{B}$ and $\vec{C}$, then

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C}).$$

[3] If $\vec{A}$ and $\vec{B}$ are parallel, then $\theta = 0$. If follows that:

$$\vec{A} \times \vec{A} = \vec{B} \times \vec{B} = 0.$$

[4] $|\vec{A}||\vec{B}| \sin\theta$ is the area of the parallelogram formed by the $\vec{A}$ and $\vec{B}$ vectors.

[5] If $\vec{A}$ and $\vec{B}$ are perpendicular, then $\theta = 90^\circ$. It follows that:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|.$$

[6] The signs are interchangeable, viz: $\vec{A} \times (-\vec{B}) = (-\vec{A}) \times \vec{B}$.

Therefore, the unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$ obey the following rules:

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0.$$
$$\hat{i}\times\hat{j} = \hat{k} : \hat{j}\times\hat{i} = -\hat{k}.$$
$$\hat{j}\times\hat{k} = \hat{i} : \hat{k}\times\hat{j} = -\hat{i}.$$
$$\hat{k}\times\hat{i} = \hat{j} : \hat{i}\times\hat{k} = -\hat{j}.$$

If we write $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$, then

$$\begin{aligned}\vec{A}\times\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x\hat{i}\times\hat{i} + A_xB_y\hat{i}\times\hat{j} + A_xB_z\hat{i}\times\hat{k} \\ &\quad + A_yB_x\hat{j}\times\hat{i} + A_yB_y\hat{j}\times\hat{j} + A_yB_z\hat{j}\times\hat{k} \\ &\quad + A_zB_x\hat{k}\times\hat{i} + A_zB_y\hat{k}\times\hat{j} + A_zB_z\hat{k}\times\hat{k} \\ &= A_xB_y\hat{k} - A_xB_z\hat{j} - A_yB_x\hat{k} + A_yB_z\hat{i} + A_zB_x\hat{j} - A_zB_y\hat{i}.\end{aligned}$$

If we collect terms in $\hat{i}$, $\hat{j}$ and $\hat{k}$, we obtain:

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}.$$

Example [1]: $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = -\hat{i} + 2\hat{j}$ are two vectors lying in the $x-y$ plane. Find $\vec{A}\times\vec{B}$.

Solution:

$$\vec{A}\times\vec{B} = (2\hat{i} + 3\hat{j})\times(-\hat{i} + 2\hat{j}) = (-2\hat{i}\times\hat{i}) + (4\hat{i}\times\hat{j}) + (-3\hat{j}\times\hat{i}) + (6\hat{j}\times\hat{j}) = 4\hat{k} + 3\hat{k} = 7\hat{k}$$

Note that the resultant vector ($7\hat{k}$) is perpendicular to the $\hat{i}, \hat{j}$ plane, which contains both vectors $\vec{A}$ and $\vec{B}$, i.e., $\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

Example [2]: What is the angle between the two vectors in problem 1?

Solution:

From the definition of the cross product, we know that $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$.

$$\therefore \sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}.$$

We know $|\vec{A}| = \sqrt{2^2 + 3^2} = 3.61$ and $|\vec{B}| = \sqrt{(-1)^2 + 2^2} = 2.24$, and in the first example, we found $|\vec{A}\times\vec{B}| = 7$.

$$\therefore \sin\theta = \frac{7}{3.61\times 2.24} = 0.866, \text{ i.e., } \theta = 60^\circ.$$

A problem for you: A force $\vec{F} = (2.00\hat{i} + 3.00\hat{j})\text{ N}$ is applied to a disk whose axis of rotation is aligned in the z-axis direction (perpendicular to $\hat{i}$ and $\hat{j}$). If the force is applied at a point whose position vector is $\vec{r} = (4.00\hat{i} + 5.00\hat{j})\text{ m}$, find (a) the torque vector $\vec{\tau}$, which is given by the relation $\vec{\tau} = \vec{r}\times\vec{F}$. (b) Find the angle between $\vec{r}$ and $\vec{F}$.

(Ans: (a) $\vec{\tau} = 2.00\hat{k}\text{ N.m}$, (b) 4.97°.)