CHAPTER 11

THE GRAVITATIONAL FIELD
(GRAVITY)

• Newton’s Law of Gravitation  
  ○ gravitational and inertial mass

• Gravitational potential energy  
  ○ satellites  
  ○ Kepler’s 3rd Law  
  ○ escape speed

• Gravitational field inside spheres*  
  ○ hollow sphere  
  ○ solid sphere

* You study using notes provided.

GRAVITATIONAL FIELD:

The groundwork for Newton’s great contribution to understanding gravity was laid by three majors players:

- Copernicus (1473-1543)
- Galileo (1564-1642)
- Kepler (1571-1630)

• shape of planetary orbits
• speed around an orbit
• relating time around orbit to distance from the Sun
A value for $G$, the \textit{universal gravitational constant}, can be obtained from measurements by Henry Cavendish (1731-1810) of the specific gravity of the Earth, as we shall show shortly. Cavendish’s value for the specific gravity of the Earth was 5.48, i.e., the density of the Earth is 5.48 times the density of water:

$$\rho_{\text{Earth}} = 5.48 \times 1000 \text{ kg/m}^3 = 5480 \text{ kg/m}^3.$$ 

So, Cavendish’s experiment allowed the first determination of the mass of the Earth; it is within 1\% of today’s \textit{accepted} value.

Consider the apple that allegedly fell in Newton’s orchard.

Newton’s 3rd Law tells us that:

$$\vec{F}_{\text{EA}} = -\vec{F}_{\text{AE}}.$$ 

Using the 2nd Law we have:

$$|m_A \vec{a}_A| = |m_E \vec{a}_E|.$$ 

But $m_A \approx 0.1 \text{ kg}$ and $m_E \approx 6 \times 10^{24} \text{ kg}$.

$$\therefore a_E \approx 1.67 \times 10^{-25} a_A.$$ 

Since near the Earth’s surface, $a_A = g$,

$$a_E = 1.64 \times 10^{-24} \text{ m/s}^2.$$ 

So, the Earth also experiences an acceleration towards the apple, but it is insignificant!
The force on the apple (i.e., its weight) is
\[ F_{EA} = G \frac{m_E m_A}{r^2} = G \frac{m_E m_A}{(R_E + h)^2}, \]
and the acceleration of the apple is
\[ a = \frac{F_{EA}}{m_A} = G \frac{m_E}{(R_E + h)^2}. \]
Very close to Earth’s surface, \( h \ll R_E \) and \( a \approx g \), so
\[ G = \frac{R_E^2 g}{m_E}. \]
where \( R_E \) is the Earth’s radius. We know \( g = 9.8 \text{m/s}^2 \), \( m_E = 6.02 \times 10^{24} \text{kg} \) and \( R_E = 6.4 \times 10^6 \text{m} \),
\[ \therefore G = 6.67 \times 10^{-11} \text{N} \cdot \text{m}^2/\text{kg}^2. \]

Re-arranging the expression for the universal gravitational constant, gives us a method for determining the acceleration at the surface of a massive object, is valid universally. So, if we know the mass \( m_p \) and radius \( r_p \) of a planet we can determine the acceleration due to gravity at the surface because \( G \) is a universal constant!

\[ g_p = G \frac{m_p}{R_p^2} \]

Here are some examples ...

Earth
mass = \( 6 \times 10^{24} \text{kg} \)
radius = \( 6378 \text{km} \)
g = 9.8 \text{m/s}^2
In deriving the expression for $g$ we have made an assumption ... that the Earth and planets are spherical!

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Mars
- mass = $6.42 \times 10^{23}$ kg
- radius = 3393 km
- $g_{\text{Mars}} = 3.7 \text{ m/s}^2 = 0.38g$

Moon
- mass = $7.35 \times 10^{22}$ kg
- radius = 1738 km
- $g_{\text{Moon}} = 1.6 \text{ m/s}^2 = 0.165g$

... it’s really a “flattened” sphere with $R_{\text{eq}} > R_p$.

\[ g = G \frac{m_{\text{E}}}{R_{\text{E}}^2} \]

\[ \therefore g_{\text{eq}} < g_p. \]

In fact, $g_p \sim 9.83 \text{ m/s}^2$ and $g_{\text{eq}} \sim 9.78 \text{ m/s}^2$. Since ones weight is $mg$, one weighs less
- up a mountain,
- in an airplane,

than at sea-level!

At 30,000 ft: $\frac{\Delta w}{w} \sim -0.3\%.$
Gravitational and inertial mass

So far, without stating it explicitly we have used the same “mass” (m) in Newton’s Law of gravitation as was defined by Newton’s 2nd Law in chapter 4. Note that in the definition of the inertial mass of an object,

\[ \text{inertial mass} = m_{\text{inert}} = \frac{F}{a}, \]

there is no mention of gravity. The mass used in Newton’s Law of gravitation is called the gravitational mass, which can be determined by measuring the attractive force exerted on it by another mass (M) a distance r away, viz:

\[ \text{gravitational mass} = m_{\text{grav}} = \frac{r^2F_{\text{Mm}}}{GM}. \]

There is no mention of acceleration in this expression. Although these are two different concepts of mass, Newton asserted that, for the same object, these masses are identical, i.e., \( m_{\text{grav}} = m_{\text{inert}} \). This is known as the principle of equivalence.

Gravitational potential energy

The acceleration of gravity at a distance \( h \) above the Earth’s surface is:

\[ g = G \frac{m_E}{r^2} = G \frac{m_E}{(h + R_E)^2}. \]

(Note: it is never zero so you are never completely weightless!!)

<table>
<thead>
<tr>
<th>Height above Earth’s surface ( h ) (×10^3 km)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>g ( (m/s^2) )</td>
<td>9.81</td>
<td>8.0</td>
<td>6.0</td>
<td>4.0</td>
<td>2.0</td>
<td>0</td>
</tr>
</tbody>
</table>

\( R_E = 6.38 \times 10^6 \text{ m} \)

\( m_E = 5.98 \times 10^{24} \text{ kg} \)
Near the Earth’s surface the gravitational potential energy of an object of mass \( m \) is

\[
U_g = mgh = mg(r - R_E),
\]

where \( U_g = 0 \) at the Earth’s surface. Far from the Earth we must take into account the fact that \( g \propto r^{-2} \). In chapter 6 we defined the change in potential energy as

\[
dU = -\vec{F} \cdot d\vec{s},
\]

where \( \vec{F} \) is a (conservative) force acting on the object and \( d\vec{s} \) is a general displacement.

For the gravitational force,

\[
dU = -F_g \cdot d\vec{s} = -(F_g \hat{r}) \cdot d\vec{s} = F_g \hat{r} \cdot d\vec{s} = G\frac{m_Em}{r^2}dr.
\]

\[
\therefore U(r) = \int S G\frac{m_Em}{r^2}dr = GmE m \int \frac{dr}{r^2} = -G\frac{m_Em}{r} + U_o,
\]

where \( U_o \) is an integration constant. We choose \( U_o = 0 \), i.e., we define the gravitational potential energy to be zero at \( r = \infty \).

We obtain a negative value for \( U(R_E) \) because we defined \( U(\infty) = 0 \). In earlier chapters we usually defined \( U(R_E) = 0 \). However, it does not cause difficulties as we normally work with changes in potential energy, i.e., \( \Delta U \), which is independent of our choice of \( U_o \).
We have
\[ U(r) = -G \frac{mEm}{r} = -G \frac{mEm}{(RE + h)} = -G \frac{mEm}{RE\left(1 + \frac{h}{RE}\right)} . \]

For small values of \( h \), \( \frac{h}{RE} \ll 1 \), then
\[
U(r) = -G \frac{mEm}{RE} \left(1 + \frac{h}{RE}\right)^{-1} = -G \frac{mEm}{RE} \left(1 - \frac{h}{RE}\right)
\]
\[= -G \frac{mEm}{RE} + G \frac{mEm}{R^2} \cdot h. \]

But \( G \frac{mEm}{RE} = U(RE) \) is constant and \( G \frac{mEm}{R^2} = g. \)

\[ \therefore U(r) = U(RE) + mgh, \]

i.e., in moving an object a height \( h \) above the Earth’s surface increases the gravitational potential energy by \( \Delta U = mgh \), providing \( h \ll RE \), a result we used in chapters 6 and 7.

Note that the weight of an object is \( Fg = mg \), i.e., \( \vec{g} \) is a vector and is called the gravitational field.

Since \( \vec{g} \) is a vector field, the resultant gravitational field of a system of masses is a vector sum of the individual contributions.

**Question 1**: Five objects, each of mass 3.00kg are equally spaced on the arc of a semicircle of radius 10.0cm. An object of mass 2.00kg is located at the center of curvature of the arc. (a) What is the gravitational force on the 2.00kg mass? (b) What is the gravitational field at the center of curvature of the arc?
(a) The force acting on \( m \) due to each of the masses \( M \) is

\[
F_x = G \frac{mM}{R^2} + G \frac{mM}{R^2} \sin 45^\circ - G \frac{mM}{R^2} \sin 45^\circ - G \frac{mM}{R^2}
\]

\[= 0,\]

and

\[
F_y = G \frac{mM}{R^2} \sin 45^\circ + G \frac{mM}{R^2} + G \frac{mM}{R^2} \sin 45^\circ
\]

\[= G \frac{mM}{R^2} \left( 1 + 2 \sin 45^\circ \right)\]

\[= 6.67 \times 10^{-11} \times 2.00 \times 3.00 \times 2.414\]

\[= 9.66 \times 10^{-8} \text{ N \ (in the \ \hat{y} \ direction)}.\]

(b) The gravitational force acting on \( m \) is \( \vec{F} = mg \).

\[\therefore \ g = \frac{F}{m} = \frac{F_x \hat{i} + F_y \hat{j}}{m} = \frac{9.66 \times 10^{-8} \hat{j}}{2.00}\]

\[= 4.83 \times 10^{-8} \hat{j} \text{ m/s}^2.\]
Question 2: A solid sphere of radius $R$ and density $\rho_o$ has its center at the origin. There is a hollow, spherical cavity of radius $r = \frac{R}{2}$ within the sphere and centered at $x = \frac{R}{2}$, as shown. Find the gravitational field at points on the $x$-axis for $|x| > R$.

We will model the system as a sphere of mass
\[ M = \frac{4}{3}\pi R^3 \rho_o \]
and a sphere of radius $R/2$ with negative mass ($-m$).

Then
\[ g(x) = g_{\text{solid}}(x) + g_{\text{cavity}}(x) \]
\[ = G \frac{M}{x^2} + G \frac{(-m)}{(x - R/2)^2} \]
\[ = \frac{G \rho_o \left( \frac{4}{3} \pi R^3 \right)}{x^2} + \frac{G \rho_o \left( \frac{-4}{3} \pi \left( \frac{R}{2} \right)^3 \right)}{(x - R/2)^2} \]
\[ = G \left( \frac{4 \pi \rho_o R^3}{3} \right) \left( \frac{1}{x^2} - \frac{1}{8(x - R/2)^2} \right). \]

When $x = R$
\[ g(R) = G \frac{2 \pi \rho_o R}{3}. \]
Question 3: An object is fired upward from the surface of the Earth with an initial speed of 4.0 km/s. Find the maximum height it achieves. Compare that height with the height it would have achieved if the gravitational field were constant.

We use conservation of energy, i.e., $K_i + U_i = K_f + U_f$.

Using the expression for gravitational potential energy, we get

$$\frac{1}{2}mv_i^2 - G\frac{m_Em}{R_E} = 0 - G\frac{m_Em}{(R_E + h)}.$$

Solving for $h$ we find

$$h = \frac{R_E}{\left(\frac{2Gm_E}{R_Ev_i^2} - 1\right)} = \frac{R_E}{\left(\frac{2gR_E}{v_i^2} - 1\right)},$$

where $g = \frac{Gm_E}{R_E^2}$ is the gravitational field at the Earth’s surface.

∴ $h = \frac{6370 \times 10^3}{\left(\frac{2 \times 9.81 \times 6370 \times 10^3}{(4.0 \times 10^3)^2} - 1\right)}$

$= 9.35 \times 10^5 \text{ m} \quad (935 \text{ km}).$
If the gravitational field \( g \) were constant, then
\[
\Delta K = -\Delta U = mgh',
\]
i.e., \( h' = \frac{v_i^2}{2g} = \frac{(4.0 \times 10^3)^2}{2 \times 9.81} = 815.5 \text{ km}. \)

In the first part we found the “correct” height was
\[
h = \frac{R_E}{\left(\frac{2gR_E}{v_i^2} - 1\right)}.
\]

But, from above, \( \frac{v_i^2}{2g} = h' \), where \( h' \) is the height for a constant gravitational field.
\[
\therefore h = \frac{R_E}{\left(\frac{R_E}{h'} - 1\right)} = \frac{h'R_E}{(R_E - h')}.
\]

Check \( h(\text{km}) = \frac{815.5 \times 6370}{(6370 - 815.5)} = 935 \text{ km}. \)

### Satellites

Consider a satellite, mass \( m \), in a circular orbit, radius \( r \), around the Earth (or a planet around the Sun). The gravitational force between the satellite and the Earth provides the centripetal acceleration.

\[
\therefore G\frac{m_E m}{r^2} = \frac{m v^2}{r}.
\]

So, in a stable orbit
\[
v^2 = \frac{Gm_E}{r}.
\]

The time for one orbit \( T = \frac{2\pi r}{v} \)
\[
\text{i.e., } v^2 = \frac{4\pi^2 r^2}{T^2} = \frac{G m_E}{r} = g'r,
\]
where \( g' \) is the gravitational field acting on the satellite.
\[ T^2 = \frac{4\pi^2}{G m_E} r^3 \] i.e., \( T^2 \propto r^3 \).

~ *Kepler’s 3rd Law of planetary motion~

Note also \[ v = \sqrt{\frac{G m_E}{r}} \]

i.e., the velocity of a satellite in a stable orbit does not depend on its mass only the mass of the central object!

Consider the Moon as the satellite around the Earth.

The speed of the Moon is

\[ v = \frac{\text{distance around orbit}}{\text{time of orbit}} = \frac{2\pi r}{T} \]

\[ = \frac{2\pi \times 3.84 \times 10^8}{27.3 \times 24 \times 60 \times 60} = 1023 \text{m/s}. \]

From above we have: \[ m_E = v^2 \frac{r}{G} \]

\[ = \frac{1023^2 \times 3.84 \times 10^8}{6.67 \times 10^{-11}} = 6.0 \times 10^{24} \text{kg}, \]

which the the same as the result we got before.

*Note:* we didn’t need to know the mass of the Moon, only its distance from the Earth and its orbital period.
Using a similar approach we can determine the mass of the Sun by thinking of the Earth as a satellite!

Look ...

\[ v = \frac{2\pi r}{T} = \sqrt{\frac{G m_S}{r}} \]

\[ 1.5 \times 10^8 \text{ km} \]

\[ v = \frac{2\pi \times 1.5 \times 10^{11}}{365 \times 24 \times 60 \times 60} = 2.99 \times 10^4 \text{ m/s}. \]

So \[ m_S = \frac{v^2 r}{G} = (2.99 \times 10^4)^2 \times \frac{1.5 \times 10^{11}}{6.67 \times 10^{-11}} \]

\[ = 2.0 \times 10^{30} \text{ kg}. \]

Question 4: If \( K \) is the kinetic energy of a satellite in a stable Earth orbit and \( U_g \) is the potential energy of the Earth-satellite system, what is the relationship between \( K \) and \( U_g \)?
We saw that the centripetal force of an Earth-satellite system is a result of the gravitational attraction between the Earth and the satellite, i.e.,

\[ F_g = G \frac{m_Em_m}{r^2} = \frac{mv^2}{r}, \]

so \( v^2 = G \frac{m_E}{r} \). ∴ \( K = \frac{1}{2} mv^2 = \frac{1}{2} G \frac{m_E m}{r} \).

But, the potential energy is \( U_g = -G \frac{m_E m}{r} \), ∴ \( K = - \frac{U_g}{2} \).

Therefore, the total mechanical energy is

\[ E = K + U_g = \frac{U_g}{2} \left( = - \frac{1}{2} G \frac{m_E m}{r} \right). \]

Note that \( E < 0 \) so the satellite is “bound”.

**Question 5:** Europa orbits Jupiter with a period of 3.55d at an average distance of \( 6.71 \times 10^8 \) m from Jupiter’s center. (a) Assuming the orbit is circular, what is the mass of Jupiter? (b) Another moon, Callisto, orbits Jupiter with a period of 16.7d. What is its average distance from Jupiter?
(a) In algebraic form, Kepler’s 3rd Law states that

$$T_e^2 = \frac{4\pi^2}{GM_J}R_e^3,$$

where $T_e$ and $R_e$ are the orbital period and the orbital radius of Europa, respectively, and $M_J$ is the mass of Jupiter.

$$\therefore M_J = \frac{4\pi^2}{G T_e^2}R_e^3 = \frac{4\pi^2 \times (6.71 \times 10^8)^3}{6.67 \times 10^{-11} \times (3.55 \times 24 \times 3600)^2} = 1.90 \times 10^{27} \text{ kg.}$$

(b) Also, $T_c^2 = \frac{4\pi^2}{GM_J}R_c^3$, where subscript $c$ refers to the moon Callisto.

$$\therefore \left(\frac{T_c}{T_e}\right)^2 = \left(\frac{R_c}{R_e}\right)^3, \text{ i.e., } R_c = R_e\sqrt[3]{\left(\frac{T_c}{T_e}\right)^2}$$

$$= 6.71 \times 10^8 \times \sqrt[3]{\left(\frac{16.7}{3.55}\right)^2} = 1.88 \times 10^9 \text{ m.}$$

Work done to put a satellite in orbit

The work done against the gravitational force in moving an object a distance $dr$ is

$$dW = \mathbf{F} \cdot d\mathbf{r} = \left(G \frac{M_E m}{r^2}\right)dr \quad (\mathbf{F} \parallel \mathbf{r}).$$

(Note, if $dr \approx h$ is very small compared with $r$, e.g., taking $r$ as the Earth’s radius (6400 km) and $h \approx$ a few meters, so $g$ is constant, then)

$$dW = m \left(G \frac{m_E}{R_E^2}\right)h = mgh,$$

a result we obtained before for the work done in lifting an object a short distance.)
Hence, the work done lifting a satellite to a height \( h \) is

\[
W = \int_{R_E}^{R_E+h} \left( G \frac{M_E m}{r^2} \right) dr = GM_E m \int_{R_E}^{R_E+h} \left( \frac{1}{r^2} \right) dr
\]

\[
= GM_E m \left[ -\frac{1}{r} \right]_{R_E}^{R_E+h} = -GM_E m \left[ \frac{1}{R_E+h} - \frac{1}{R_E} \right]
\]

\[
= -GM_E m \left[ \frac{R_E - (R_E + h)}{R_E(R_E + h)} \right] = m \left( \frac{GM_E}{R_E^2} \right) \frac{hR_E}{(R_E + h)}
\]

\[
= mg \left( \frac{hR_E}{(R_E + h)} \right).
\]

By the work-energy theorem, the launch speed required for the satellite to reach a height \( h \) is given by

\[
\frac{1}{2} mv^2 = mg \left( \frac{hR_E}{(R_E + h)} \right),
\]

i.e.,

\[
v = \sqrt{2g \left( \frac{hR_E}{(R_E + h)} \right)}.
\]

The escape speed \( v_e \) is defined as the minimum speed required for an object to “escape” the Earth’s gravitational field. Then

\[
K_f = 0; U(\infty) = 0
\]

\[
K_i + U_i = K_f + U_f,
\]

i.e.,

\[
\frac{1}{2} mv_e^2 - U(R_E) = 0.
\]

\[
K_i = \frac{1}{2} mv_i^2; U(R_E)
\]

\[
\therefore \frac{1}{2} mv_e^2 = \frac{GM_E m}{R_E},
\]

so,

\[
v_e = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2gR_E}.
\]

For the Earth, we find

\[
v_e = \sqrt{2 \times 9.81 \times 6370 \times 10^3} = 1.12 \times 10^4 \text{ m/s}
\]

\[= 11.2 \text{ km/s}.
\]

For an object launched from Earth,

- if \( v_e < 11.2 \text{ km/s} \) the object will be “bound” and will orbit the Earth in a circle or ellipse,
- if \( v_e > 11.2 \text{ km/s} \) the object will be “unbound” and will follow a hyperbolic path and leave the Earth and never return.
Question 6: One way of thinking about a black hole is to consider a spherical object whose density is so large that the escape speed at the surface is greater than the speed of light, c. The critical radius for the formation of a black hole is called the Schwarzschild radius, $R_S$. (a) Show that $R_S = \frac{2GM}{c^2}$, where $M$ is the mass of the black hole. (b) Calculate the Schwarzschild radius for a black hole whose mass is equal to 10 solar masses.

(a) The escape speed is given by

$$v_e = \sqrt{\frac{2GM}{R}}.$$  

For a black hole, $v_e = c$ and $R = R_S$, i.e., $c = \sqrt{\frac{2GM}{R_S}} \Rightarrow R_S = \frac{2GM}{c^2}$.

(b) If $M = 10M_{\text{Sun}} = 10 \times 1.99 \times 10^{30} = 1.99 \times 10^{31}$ kg,

$$\therefore R_S = \frac{2 \times 6.67 \times 10^{11} \times 1.99 \times 10^{31}}{(3 \times 10^8)^2}$$

$$= 2.95 \times 10^4 \text{ m} \quad (29.5 \text{ km}).$$
Gravitational field inside a hollow sphere

The gravitational field inside a hollow sphere is zero, i.e.,
\[ \frac{r}{g} = 0 \quad (r < R), \]
We can show this result by considering a small (point) mass \( m_0 \) situated at some point inside the sphere. If the distance from the mass to points on the shell diametrically opposite are \( r_1 \) and \( r_2 \), then the areas \( A_1 \) and \( A_2 \) and the volumes \( V_1 = A_1 \Delta r \) and \( V_2 = A_2 \Delta r \) are proportional to \( r_1^2 \) and \( r_2^2 \), respectively. If the density of the shell is \( \rho \), then
\[
\frac{m_1}{\eta^2} = \frac{m_2}{\eta^2} \quad \text{and} \quad G \frac{m_1 m_0}{\eta^2} = G \frac{m_2 m_0}{\eta^2},
\]
i.e.,
\[
\frac{m_1}{\eta^2} = \frac{m_2}{\eta^2} \quad \text{and} \quad G \frac{m_1 m_0}{\eta^2} = G \frac{m_2 m_0}{\eta^2}.
\]
Thus, the net force on \( m_0 \) is zero. That argument can be made for all diametrically opposite areas \( A_1 \) and \( A_2 \) over the sphere. So, the gravitational field inside a hollow sphere is zero.

In fact, the result is true for any thickness of shell. We can apply the previous result by dividing the shell into a continuum of concentric shells and summing the fields at the the point of interest. Thus,
\[ \frac{r}{g} = 0, \]
inside all hollow spheres.

Gravitational field inside a solid sphere

To determine the gravitational field inside a solid sphere, a distance \( r \) from the center, we use the previous result. The mass of the part of the sphere outside \( r \) makes no contribution to the gravitational field at or inside \( r \). Only the mass \( M' \) inside radius \( r \) contributes to the gravitational field at \( r \). This mass produces a field equal to that of a point mass \( M' \) at the center of the sphere. If \( M \) is the mass of the sphere, then
\[ M' = \frac{4}{3} \pi \frac{r^3}{R^3} M = \frac{r^3}{R^3} M. \]

Then, using an earlier expression for \( g \), we find
\[ |g(r \leq R)| = G \frac{M'}{r^2} = \frac{GM}{R^3} r, \]
inside the sphere.

The magnitude of the gravitational field is zero at the origin and increases linearly with \( r \). But, as before, it is directed towards the center of the sphere.

The stipulation for this result is that the sphere is of uniform mass density.

**Question 7:** Agamemnon is a planet of radius 4850km and density 5500kg/m³. A straight shaft is drilled completely through the planet Agamemnon. If a stone is dropped from the top of the shaft, what would its speed be (a) at the center, and (b) at a distance of 2500km above the center? (c) Would the stone reach the opening at the other end of the shaft?
(a) If we let a mass \( m \) fall down the shaft, to a radius \( r \), the work done by the gravitational field \( \Delta W \) is

\[
\Delta W = \int_{R}^{r} F_g \cdot dr = \int_{R}^{r} mg_r dr,
\]

where \( g_r \) is the gravitational field at radius \( r \). But

\[
g_r = \frac{GM'}{r^2} = \frac{GM}{R^3} r,
\]

where \( M' \) is the mass contained within the sphere of radius \( r \).

\[
\therefore \Delta W = -\int_{R}^{r} \frac{GMm}{R^3} r dr = \frac{GMm}{2R^3} \left( r^2 - R^2 \right)
\]

But \( M = \frac{4}{3} \pi R^3 \rho = \frac{4}{3} \pi (4850 \times 10^3)^3 \times 5500 = 2.63 \times 10^{24} \text{ kg} \),

and from the work-energy theorem, \( \Delta W = \Delta K \).

With \( R = 4850 \text{ km} \) and \( r = 0 \),

\[
\Delta W = \Delta K = \frac{GM}{2R} m = \frac{6.67 \times 10^{-11} \times 2.63 \times 10^{24}}{2 \times 4850 \times 10^3} \text{ m} \ (J).
\]

i.e., \( \frac{1}{2} mv^2 = 1.81 \times 10^7 \text{ m} \).

\[
\therefore v = \sqrt{2 \times 1.81 \times 10^7} = 6014 \text{ m/s}.
\]

(b) With \( R = 4850 \text{ km} \) and \( r = 2500 \text{ km} \),

\[
\Delta W = \Delta K = 1.33 \times 10^7 \text{ m} \ (J).
\]

\[
\therefore v = \sqrt{2 \times 1.33 \times 10^7} = 5154 \text{ m/s}.
\]

(c) Yes, the stone would just reach the opposite opening. Since the scenario is symmetrical, its speed at the opening would be zero, having done \( \Delta W = 1.81 \times 10^7 \text{ m Joules of work against the gravitational field.} \)
Question 8: When farthest from Earth, i.e., at apogee, the Moon-Earth distance is 406,400km. When closest to Earth, i.e., at perigee, the Moon-Earth distance is 357,600km. If the mass of the Earth is $5.89 \times 10^{24}$ kg, what are the orbital speeds of the Moon at apogee and perigee?

In the previous chapter we saw that providing there are no external forces/torques, then the angular momentum of a system like the Earth-Moon is conserved. The angular momentum of the Moon is $L = r \times m v$.

But at perigee and apogee $r \perp v$, so $L = mv r$.

∴ $mv_a r_a = mv_p r_p$, i.e., $v_a = v_p \frac{r_p}{r_a}$.

Also, mechanical energy is conserved, i.e.,

$$\frac{1}{2} mv_a^2 - G \frac{M_E m}{r_a} = \frac{1}{2} mv_p^2 - G \frac{M_E m}{r_p}.$$  

∴ $v_a^2 - 2G \frac{M_E}{r_a} = v_p^2 - 2G \frac{M_E}{r_p}$.

Hence $v_p^2 \left( \frac{r_p}{r_a} \right)^2 - 2G \frac{M_E}{r_a} = v_p^2 - 2G \frac{M_E}{r_p}$,

i.e., $v_p^2 \left( 1 - \frac{r_p^2}{r_a^2} \right) = 2GM_E \left( \frac{1}{r_p} - \frac{1}{r_a} \right)$. 
\[
\therefore \, v_p^2 \left( \frac{r_a^2 - r_p^2}{r_a^2} \right) = 2GME \left( \frac{r_a - r_p}{r_ar_p} \right)
\]

so \[v_p^2 \left( \frac{r_a + r_p}{r_a} \right) = \frac{2GM_E}{r_p}\]
i.e., \[v_p = \sqrt{\frac{2GM_E r_a}{r_p(r_a + r_p)}}\]

\[
= \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times 4.064 \times 10^8}{3.576 \times 10^8 \times \left( 4.064 \times 10^8 + 3.576 \times 10^8 \right)}}
\]

\[= 1090 \text{m/s} \quad (1.090 \text{km/s}).\]

\[
\therefore \, v_a = v_p \frac{r_p}{r_a} = 1090 \times \frac{3.576 \times 10^8}{4.064 \times 10^8}
\]

\[= 959 \text{m/s} \quad (0.959 \text{km/s}).\]